

VECTOR-VALUED EXTENSIONS FOR SINGULAR BILINEAR OPERATORS AND APPLICATIONS

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The problems presented in this thesis were motivated by the study of a Rubio de Francia operator for iterated Fourier integrals associated to arbitrary intervals. This further led to vector-valued estimates for the bilinear Hilbert transform *BHT*. The vector spaces can be iterated l^p or L^p spaces, and whenever all these are locally in L^2 , we recover the *BHT* range. This is illustrated in Chapter 4.

The methods of the proof apply for paraproducts as well, as seen in Chapter 5. We prove boundedness of vector-valued paraproducts, within the same range as scalar paraproducts.

In Chapter 6, we present a few consequences: the boundedness of the initial Rubio de Francia operator for iterated Fourier integrals, the boundedness of tensor products of n paraproducts and one *BHT* in L^p spaces, and new estimates for tensor products of bilinear operators in L^p spaces with mixed norms. Since paraproducts act as mollifiers for products of functions, possibly the most important application is a new Leibniz rule in mixed norm L^p spaces.

A Rubio de Francia theorem for paraproducts is described in Chapter 3. The approach is completely different from the more abstract vector-valued method, and it is an instance where maximal paraproducts appear.

Finally, in Chapter 7 we employ our methods for re-proving vector-valued Carleson operator estimates, as well as estimates for the square function. As opposed to the Calderón-Zygmund decomposition which yields $L^1 \rightarrow L^{1,\infty}$ esti-

mates, our “localization” method is useful for proving $L^p \rightarrow L^p$, when $p \geq 2$. In both cases, the general result follows by duality.

BIOGRAPHICAL SKETCH

Cristina Benea was born in Bucharest, a couple of years before the fall of the communist regime, at a time when only a few dared to hope for a change. She grew up with her grandparents, in a village 300km away from Bucharest. Here she attended first and second grades, before she moved to the capital city, where she lived and studied until starting grad school at Cornell University.

To my grandparents.

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CHAPTER 1

INTRODUCTION

While studying mixed L^p AKNS systems, the following operator, resembling Rubio de Francia's Square Function, appeared in a natural way:

$$T_r(f, g)(x) = \left(\sum_{k=1}^N \left| \int_{a_k < \xi_1 < \xi_2 < b_k} \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^r \right)^{\frac{1}{r}}. \quad (1.1)$$

Here $r \geq 1$, and the intervals $[a_k, b_k]$ are arbitrary. The similarity with Rubio de Francia's operator

$$F \mapsto \left(\sum_{k=1}^N \left| \int_{a_k < \xi < b_k} \hat{F}(\xi) e^{2\pi i x \xi} d\xi \right|^2 \right)^{\frac{1}{2}}$$

consists in the taking of Fourier projections associated with arbitrary intervals. For this reason, we refer to T_r as a *Rubio de Francia operator for iterated Fourier integrals*. The condition $a_k < \xi_1 < \xi_2 < b_k$ points towards a vector-valued *Bilinear Hilbert Transform (BHT in short)*, where:

$$BHT(f, g)(x) = \int_{\xi_1 < \xi_2} \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2.$$

This operator is interesting in many respects: it is an example of a bilinear operator which is not a multi-linear Calderón-Zygmund operator, it is modulation-invariant (in the sense that $|BHT(e^{2\pi i a x} \cdot f, e^{2\pi i a x} \cdot g)| = |e^{2\pi i 2a x} BHT(f, g)|$), its symbol $\text{sgn}(\xi_1 - \xi_2)$ is singular along a line. So far, it is known that $BHT : L^p \times L^q \rightarrow L^s$ for any $1 < p, q \leq \infty$, given that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and $\frac{2}{3} < s < \infty$. Whether this is true for $\frac{1}{2} < s \leq \frac{2}{3}$ is still an open problem.

Returning to our initial motivation, we define AKNS systems as systems of differential equations of the form

$$u' = i\lambda Du + Au \quad (1.2)$$

where $u = [u_1, \dots, u_n]'$ is a vector valued function defined on \mathbb{R} , D is a diagonal $n \times n$ matrix with real and distinct entries d_1, d_2, \dots, d_n , and $A = (a_{jk}(\cdot))_{j,k=1}^n$ is a matrix valued function defined on \mathbb{R} , and so that $a_{jj} \equiv 0$ for all $1 \leq j \leq n$.

The so-called *spectral parameter* λ becomes the input of our function, and we are interested in L^p estimates for u^λ . For example, we want to know if

$$\|u_j^\lambda\|_\infty < \infty \quad \text{for a. e. } \lambda \quad \text{and all } 1 \leq j \leq n,$$

under the weakest possible assumptions; we only require the entries of the potential matrix A to be integrable in some L^p spaces:

$$a_{jk}(\cdot) \in L^{p_{jk}}(\mathbb{R}), \quad \text{for all } 1 \leq j, k \leq n, j \neq k.$$

In the case of an upper triangular matrix A , this problem reduces to estimating in L^p expressions of the form

$$\int_{x_1 < \dots < x_n < y} f_1(x_1) \dots f_n(x_n) e^{i\lambda(\alpha_1 x_1 + \dots + \alpha_n x_n)} dx_1 \dots dx_n. \quad (1.3)$$

Define

$$\tilde{C}_n^\alpha(f_1, f_2, \dots, f_n)(\lambda) := \sup_y \int_{x_1 < \dots < x_n < y} f_1(x_1) \dots f_n(x_n) e^{i\lambda(\alpha_1 x_1 + \dots + \alpha_n x_n)} dx_1 \dots dx_n$$

where for all k , $\alpha_k \neq 0$, and the functions f_k are entries of the original matrix A (and so $f_k \in L^{p_k}$). It was proved by Christ and Kiselev [18], [19], [17] that \tilde{C}_n^α is a bounded operator:

$$\|\tilde{C}_n^\alpha(f_1, \dots, f_n)\|_{s_n} \lesssim \prod_{k=1}^n \|f_k\|_{p_k}$$

for $1 \leq p_k < 2$, and $\frac{1}{s_n} = \frac{1}{p'_1} + \dots + \frac{1}{p'_n}$. They only deal with the case $f_1 = \dots = f_n = f$, and hence $p_1 = \dots = p_n$, but the ideas of the proof for the more general case are the same.

On the other hand, if we regard the entries f_k as the Fourier transforms of some other functions h_k , the previous expression becomes equivalent to

$$\sup_N \left| \int_{x_1 < \dots < x_n < N} \hat{h}_1(x_1) \dots \hat{h}_n(x_n) e^{i\lambda(\alpha_1 x_1 + \dots + \alpha_n x_n)} dx_1 \dots dx_n \right| \quad (1.4)$$

denoted $C_n^\alpha(h_1, \dots, h_n)(\lambda)$. For $n = 1$, this is exactly the Carleson operator, while $n = 2$ corresponds to the Bi-Carleson operator of [15], both of which were proved to be bounded operators (with the remark that for the Bi-Carleson, the α_k s need to satisfy some non-degeneracy condition):

$$\|C_2^\alpha(h_1, h_2)\|_{s_2} \lesssim \|h_1\|_{p_1} \|h_2\|_{p_2}$$

for $1 < p_1, p_2 \leq \infty$, $\frac{1}{s_2} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{2}{3} < s_2 < \infty$.

The upper-triangular AKNS problem is fully answered for entries in L^{p_i} spaces with $1 \leq p_i < 2$, by estimating \tilde{C}_n^α , and partially for entries in L^{p_i} spaces with $1 \leq p_i \leq 2$, by estimating C_n^α . Questions regarding iterated integrals, and especially iterated Fourier integrals turned out to be related to several other multilinear operators studied in time-frequency analysis as presented in [16], [21].

We therefore became interested in studying the mixed problem:

$$\sup_y \int_R f_{11}(x_{11}) \dots f_{1n_1}(x_{1n_1}) \hat{g}_{11}(\xi_{11}) \dots \hat{g}_{1m_1}(\xi_{1m_1}) \dots f_{21}(x_{21}) \dots f_{2n_2}(x_{2n_2}) \hat{g}_{11}(\xi_{11}) \dots \hat{g}_{lm_1}(\xi_{lm_1}) dx$$

where $R = \{x_{11} < \dots x_{1n_1} < \xi_{11} < \dots \xi_{1m_1} < \dots < x_{21} < \dots x_{2n_2} < \dots \xi_{l1} < \xi_{lm_l}\}$. This is a hybrid of the C_n^α s and \tilde{C}_m^α s.

The simplest of these operators, where sup is dropped, is

$$M(f_1, f_2, g)(\xi) = \int_{x_1 < x_2 < x_3} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3, \quad (1.5)$$

where $f_1 \in L^{p_1}$, $f_2 \in L^{p_2}$, $1 < p_1, p_2 < \infty$, and $g \in L^p$ with $1 < p < 2$.

The techniques used by Christ and Kiselev for proving the boundedness of \tilde{C}_n^α are based on a dyadic filtration associated to one of the functions. This involves a structure on \mathbb{R} similar to that of the dyadic mesh: on every level of the filtration, one has a partition of \mathbb{R} , and passing to the next level of the filtration means refining the previous partition. We want to use g in order to obtain this structure and for simplicity we assume $\|g\|_p = 1$. Define the function

$$\varphi(x) = \int_{-\infty}^x |g(y)|^p dy.$$

Its image is the unit interval $[0, 1]$, and the filtration will consist of pre-images through φ of the collection \mathcal{D} of dyadic intervals in $[0, 1]$. Because φ is increasing, whenever $x_2 < x_3$ we have $0 \leq \varphi(x_2) \leq \varphi(x_3) \leq 1$. Hence there exists a unique dyadic interval $\omega \subset [0, 1]$ so that $\varphi(x_2)$ is contained in the left half of ω , which we denote ω_L , while $\varphi(x_3)$ is contained in the right half ω_R . To simplify notation, we identify $\varphi^{-1}(\omega)$ with ω .

Then the operator M can be written as

$$\begin{aligned} & \sum_{\omega \in \mathcal{D}} \int_{\substack{x_1 < x_2 \\ x_2 \in \omega_L, x_3 \in \omega_R}} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3 = \\ & = \sum_{\omega} \int_{\substack{x_1 < x_2, x_1, x_2 \in \omega_L \\ x_3 \in \omega_R}} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3 + \end{aligned} \quad (1.6)$$

$$+ \sum_{\omega} \int_{\substack{x_1 < L(\omega_L), x_2 \in \omega_L \\ x_3 \in \omega_R}} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3. \quad (1.7)$$

Here $L(\omega_L)$ denotes the left endpoint of the interval ω_L . We call the operator in (1.6) M_1 and the one in (1.7) M_2 . The first term M_1 accounts for the occurrence of arbitrary intervals (they are in fact $\varphi^{-1}(\omega_L)$), and this combined with Hölder's inequality motivates the initial operator

$$T_r(f, g)(x) = \left(\sum_{k=1}^N \left| \int_{a_k < \xi_1 < \xi_2 < b_k} \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^r \right)^{\frac{1}{r}}.$$

In section 6.1 we will show how both M_1 and M_2 are bounded operators:

$$M_1 : L^{p_1} \times L^{p_2} \times L^p \rightarrow L^q \quad \text{provided } 1 < p < 2 \text{ and } \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'} = \frac{1}{q} \quad \text{while}$$

$$M_2 : L^{p_1} \times L^{p_2} \times L^p \rightarrow L^q \quad \text{provided } 1 < p < 2, \frac{1}{p_2} + \frac{1}{p'} < 1 \text{ and } \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'} = \frac{1}{q}.$$

Overall, their sum will be a bounded operator from $L^{p_1} \times L^{p_2} \times L^p$ to L^q provided $1 < p < 2, \frac{1}{p_2} + \frac{1}{p'} < 1$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'} = \frac{1}{q}$.

This was the initial motivation for T_r and vector-valued *BHT* operators. In [14], Silva gives a partial results for the boundedness of vector-valued *BHT*; namely, for $\frac{4}{3} < r < 4$,

$$\left\| \left(\sum_k |BHT(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \cdot \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q, \quad (1.8)$$

where $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$. The range of the above operator is most often smaller than that of *BHT*, and depends implicitly on r_1, r_2 , and r .

Even though it is not obvious from the formulation, these results extend to the case when one of the r_1, r_2 , and r is in the interval $\left(\frac{4}{3}, 4\right)$, and this gives an answer for the boundedness of T_1 , but not for T_r in general. One of the main results of this thesis is the extension of estimates such as (1.8) for any tuple (r_1, r_2, r) satisfying $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$. Moreover, we give an explicit description of the range. Many other applications inspired by the method of the proof follow.

CHAPTER 2

PRELIMINARIES

2.1 Maximal Operators, Singular Integrals, Square Functions

We will be using the following notation: $A \lesssim B$ for $A \leq C \cdot B$, where C is some constant, independent on the quantities A and B . $A \ll B$ means that A is much, much smaller than B . $\langle t \rangle := \sqrt{1 + t^2}$, the so-called *Japanese bracket*, ensures that the quantity $\langle t \rangle$ is comparable to $|t|$, but it is never 0.

A *dyadic interval* is one of the form $[2^k n, 2^k(n+1)]$, where $k, n \in \mathbb{Z}$. Any two dyadic intervals are either disjoint or one of them is contained inside the other one. The mesh of dyadic intervals is denoted by \mathcal{D} .

Definition 1. Let $\{\phi_I\}_I$ be a family of bump functions associated to the collection of dyadic intervals \mathcal{I} .

We say that $\{\phi_I\}_I$ is *lacunary* if each $\hat{\phi}_I$ is supported on some ω of length $\sim |I|^{-1}$, and so that $\text{dist}(\omega, 0) \sim |\omega|$. That is,

$$\text{supp } \hat{\phi}_I \subseteq \left[-\frac{4}{|I|}, -\frac{1}{4|I|}\right] \cup \left[\frac{1}{4|I|}, \frac{4}{|I|}\right].$$

A non-lacunary family of bump functions $\{\phi_I\}_I$ is so that

$$\text{supp } \hat{\phi}_I \subseteq \left[-\frac{4}{|I|}, \frac{4}{|I|}\right].$$

For a lacunary family we adopt the notation $\{\psi_I\}_I$, while for a non-lacunary family we prefer $\{\varphi_I\}_I$.

For any $0 < p < \infty$, let

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} \quad (2.1)$$

and $L^p := \{f : \mathbb{R} \rightarrow \mathbb{C} : \|f\|_p < \infty\}$. If $1 \leq p \leq \infty$, L^p is a Banach space and $\|\cdot\|_p$ is a norm. For $0 < p < 1$, L^p is a quasi-Banach space and $\|\cdot\|_p$ is a quasi-norm.

$$\|f\|_\infty := \sup_E \sup_{x \in \mathbb{R} \setminus E} |f(x)|,$$

where E ranges over sets of measure 0. $L^\infty(\mathbb{R}, \|\cdot\|_\infty)$ is a Banach space, and it is the dual of $L^1(\mathbb{R})$. However, its dual is not $L^1(\mathbb{R})$.

Later on, we will prove some results in L^p spaces with mixed norms. These are $L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}$ with the norm:

$$\|f\|_{p_1 p_2 \dots p_M} := \|f\|_{L_{x_1}^{p_1} \dots L_{x_M}^{p_M}} := \| \|f(x_1, \dots, x_M)\|_{L_{x_1}^{p_1}} \|_{L_{x_M}^{p_M}}.$$

So the left-most index corresponds to the inner-most norm, in our notation. These spaces and appropriate interpolation methods were introduced in [1], and they appear naturally in the study of differential equations which are space and time dependent.

Definition 2. The distribution of a function $f : \mathbb{R} \rightarrow \mathbb{C}$ is defined to be

$$d_f(\lambda) := |\{x : |f(x)| > \lambda\}|.$$

Using the distribution function, we can define *weak L^p spaces*. For $0 < p < \infty$,

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p} = \sup_{\lambda > 0} \lambda d_f(\lambda)^{1/p}. \quad (2.2)$$

Then $L^{p,\infty} := \{f : \|f\|_{p,\infty} < \infty\}$ is a quasi-Banach space for any $0 < p < \infty$. Also, $L^{\infty,\infty}$ coincides with the regular L^∞ space.

Remark 1. (i) For any $0 < p < \infty$,

$$|\{x : |f(x)| > \lambda\}| \leq \frac{1}{\lambda^p} \|f\|_p^p. \quad (2.3)$$

This is sometimes referred to as Chebyshev's inequality. An immediate consequence is that $L^p \subseteq L^{p,\infty}$.

(ii) Due to Fubini's theorem, there is a way of expressing $\|f\|_p$ using the distribution function:

$$\|f\|_p^p := p \int_0^\infty \lambda^{p-1} d_f(\lambda) d\lambda.$$

(iii) Weak L^p spaces are suitable for interpolation, as seen in the Marcinkiewicz interpolation theorem. Furthermore, many of the singular integral operator in harmonic analysis are not bounded from L^1 to L^1 , but rather from L^1 to $L^{1,\infty}$.

Definition 3. The maximal operator is defined as

$$\mathcal{M}f(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy, \quad (2.4)$$

where supremum ranges over all open balls $B \subset \mathbb{R}^d$, containing x .

Then $\mathcal{M} : L^p \rightarrow L^p$ for any $1 < p \leq \infty$. $\mathcal{M} : L^1 \not\rightarrow L^1$, but $\mathcal{M} : L^1 \rightarrow L^{1,\infty}$.

Remark 2. Alternatively, one can define a centered maximal operator

$$\tilde{\mathcal{M}}f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

The two definitions are equivalent, in the sense that $\tilde{\mathcal{M}}f(x) \leq \mathcal{M}f(x) \leq c\tilde{\mathcal{M}}f(x)$ for any $x \in \mathbb{R}^d$.

Definition 4. Let Φ be a Schwartz function with $\int_{\mathbb{R}} \Phi(x) dx = 1$, and denote $\Phi_k(x) = 2^k \Phi(2^k x)$. Then

$$\sup_{k \in \mathbb{Z}} |f * \Phi_k(x)| \leq C \mathcal{M}f(x).$$

The LHS is an example of a maximal operator obtained by convoluting with an approximation of the identity.

Remark 3. We can relax the assumptions on Φ . Following [28], it is sufficient for Φ to be integrable, decrease “at a sufficient uniform rate at infinity”, and $\int \Phi dx = 1$. These conditions will guarantee $L^p \rightarrow L^p$ boundedness of the convolution maximal

operator, for $1 < p < \infty$. The $L^1 \rightarrow L^{1,\infty}$ boundedness requires Φ to satisfy a Dini type condition, but we will not dwell on this aspect.

In section 4.13, we will deal with functions Φ that decay like $\frac{1}{(1+|x|)^{1+\alpha}}$ at infinity, where $0 < \alpha < 1$. In that case, $f \mapsto \sup_k |f * \Phi_k|$ is bounded on L^p , for $1 < p < \infty$.

Another extremely important variant of the maximal operator is the *shifted maximal operator*

$$\mathcal{M}^n f(x) := \sup_{x \in I} \frac{1}{|I|} \int_{\mathbb{R}} |f(y)| \tilde{\chi}_{I_n}(y) dy,$$

where $I_n := I + n|I|$ is the shift of the interval I n units to the right (if n is positive), or n units to the left (if n is negative). Then \mathcal{M}^n is a bounded operator, and the fact that the intervals are shifted is reflected in the operatorial norm:

Theorem 1 (Thm. 4.5 in [6]). *For any integer n , the shifted maximal operator \mathcal{M}^n is bounded on every L^p space, with $1 < p < \infty$. \mathcal{M}^n is also bounded from L^1 to $L^{1,\infty}$, with operatorial norm of the type $O(\log \langle n \rangle)$:*

$$\|\mathcal{M}^n(f)\|_p \lesssim \log \langle n \rangle \|f\|_p, \quad \text{and} \quad \|\mathcal{M}^n(f)\|_{1,\infty} \lesssim \log \langle n \rangle \|f\|_1.$$

2.1.1 Square Functions

We begin by introducing a very special partition of unity. Let ϕ be a Schwartz function so that $\hat{\phi}(\xi) \equiv 1$ on $|\xi| \leq \frac{1}{2}$, and $\hat{\phi}$ is supported in $|\xi| \leq 1$. First, note that

$$\hat{\psi}(\xi) := \hat{\phi}\left(\frac{\xi}{2}\right) - \hat{\phi}(\xi)$$

is supported in the region $\frac{1}{2} \leq |\xi| \leq 2$. Define ψ_k so that $\hat{\psi}_k(\xi) = \hat{\psi}\left(\frac{\xi}{2^k}\right)$; then

$$\hat{\psi}_k(\xi) = \hat{\phi}\left(\frac{\xi}{2^k}\right) - \hat{\phi}\left(\frac{\xi}{2^{k-1}}\right)$$

is supported in the shell $2^{k-1} \leq |\xi| \leq 2^{k+1}$, and most importantly,

$$\sum_k \hat{\psi}_k(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (2.5)$$

This implies that

$$f = \sum_k f * \psi_k(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Notation 1. The map $f \mapsto f * \psi_k$, denoted P_k , is the Fourier projection onto the shell $2^{k-1} \leq |\xi| \leq 2^{k+1}$. Similarly, $f \mapsto f * \phi_k$, denoted Q_k , is the Fourier projection onto the ball $|\xi| \leq 2^k$.

Definition 5. The Littlewood-Paley square function is defined by

$$Sf(x) = \left(\sum_{k \in \mathbb{Z}} |f * \psi_k(x)|^2 \right)^{1/2} = \left(\sum_{k \in \mathbb{Z}} |P_k f(x)|^2 \right)^{1/2}.$$

Theorem 2. For any $1 < p < \infty$, there exists a constant $C > 0$ so that

$$C^{-1} \|f\|_{L^p(\mathbb{R}^d)} \leq \|Sf\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

In other words, the L^p norm of Sf and the L^p norm of f are equivalent.

Also, $S : L^1 \rightarrow L^{1,\infty}$.

Though apparently similar, the square function with sharp cutoffs

$$S_{\text{sharp}}(f)(x) = \left(\sum_{k \in \mathbb{Z}} |f * \mathbf{1}_{\{2^{k-1} \leq |\xi| < 2^k\}}|^2 \right)^{1/2}$$

is bounded on $L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ only in the case $d = 1$. This fails for $d \geq 2$ because the ball multiplier does not define a bounded operator, as shown in [8].

However, in the case $d = 1$, $S_{\text{sharp}} : L^p \rightarrow L^p$ and $S_{\text{sharp}} : L^1 \rightarrow L^{1,\infty}$. We will be mainly dealing with the one dimensional case. For $d = 1$, there exists a discretized version of this theorem, that will be used throughout the thesis.

Theorem 3. Let \mathcal{I} be a family of dyadic intervals to which we associate a family of functions $(\varphi_I)_{I \in \mathcal{I}}$. Each φ_I has the property that its Fourier transform $\hat{\varphi}_I$ is supported on $\pm \left[\frac{1}{|I|}, \frac{2}{|I|} \right]$, and is $\equiv |I|^{1/2}$ on $\frac{9}{10} \left[\frac{1}{|I|}, \frac{2}{|I|} \right]$. Define

$$\mathcal{S}(f)(x) := \left(\sum_{I \in \mathcal{I}} \frac{|\langle f, \varphi_I \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{1/2}.$$

Then $\mathcal{S} : L^p \rightarrow L^p$ for any $1 < p < \infty$, and $L^1 \rightarrow L^{1,\infty}$, with the operatorial norm independent on the collection of dyadic intervals \mathcal{I} .

Other related discrete square functions will be presented in section 7.1.

There is also a *shifted square function operator*

$$\mathcal{S}^n f(x) := \left(\sum_{I \in \mathcal{I}} \frac{|\langle f, \phi_{I_n} \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{1/2},$$

for which we have:

Theorem 4 (Thm. 4.6 in [6]). For any integer $n \in \mathbb{Z}$, we have

$$\|\mathcal{S}^n(f)\|_p \lesssim \log \langle n \rangle \|f\|_p, \quad \text{and} \quad \|\mathcal{S}^n(f)\|_{1,\infty} \lesssim \log \langle n \rangle \|f\|_1.$$

2.1.2 Rubio de Francia's Square Function

A natural question is whether something similar to the Littlewood Paley sharp square function holds for arbitrary projections in frequency, not only $[2^{k-1}, 2^k]$. More exactly, let $(I_k)_k = ([a_k, b_k])_k$ be an arbitrary family of intervals in \mathbb{R} . We define *Rubio de Francia's square function* by

$$RF(f)(x) = \left(\sum_k |f * \check{\mathbf{1}}_{I_k}(x)|^2 \right)^{1/2} = \left(\sum_k |P_{I_k} f|^2 \right)^{1/2},$$

where $P_{I_k}f = \mathcal{F}^{-1}(\hat{f} \cdot \mathbf{1}_{I_k})$ is the Fourier projection onto the interval I_k . For the case of arbitrary intervals, we have the following, apparently dual(at least in the case $1 < q$), results:

Theorem 5 (Rubio de Francia, [9]). *For any $2 \leq p < \infty$,*

$$\left\| \left(\sum_k |P_{I_k}f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p,$$

with a constant C independent on the family of arbitrary intervals.

Theorem 6 (Bourgain, [3]). *For any $1 \leq q \leq 2$, and any function f with \hat{f} supported on $\cup_k I_k$,*

$$\|f\|_q \leq C \left\| \left(\sum_k |P_{I_k}f|^2 \right)^{1/2} \right\|_q,$$

where the constant appearing in the above expression is independent on the family of arbitrary intervals.

In section 7.2, we provide a new proof for theorem 5. Results similar to the case $q = 1$ of theorem 6 appear in section 6.2.

While the l^2 setting is best for describing orthogonality properties, there is an l^ν -variant for Rubio de Francia's square function:

$$RF_\nu f(x) = \left(\sum_k |f * \mathbf{1}_{I_k}(x)|^\nu \right)^{1/\nu} = \left(\sum_k |P_{I_k}f|^\nu \right)^{1/\nu}$$

Theorem 7 (Rubio de Francia, [9]). *Let $2 \leq \nu < \infty$. Then*

$$\|RF_\nu f\|_p \leq C \|f\|_p,$$

for any $1 < p < \infty$ with the property that $\frac{1}{\nu} + \frac{1}{p} < 1$.

This is obtained by interpolation between the case $\nu = 2$ and $\nu = \infty$. The latter one follows from Carleson's theorem, stating that

$$f \mapsto C_{\mathbb{R}}f(x) = \sup_N \left| \int_{\xi < N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

is a bounded operator from L^p to L^p , for any $1 < p < \infty$.

2.1.3 Singular Integrals Operators

The first example of a singular integral operator is the *Hilbert Transform* defined by

$$Hf(x) := p.v. \int_{\mathbb{R}} f(x-y) \frac{1}{y} dy := \lim_{\epsilon \rightarrow 0} \int_{|y| > \epsilon} f(x-y) \frac{1}{y} dy \quad (2.6)$$

Proposition 1. $H : L^p \rightarrow L^p$ and $H : L^1 \rightarrow L^{1,\infty}$.

This can be generalized to singular integral operators of convolution type:

$$T(f)(x) = p.v. \int_{\mathbb{R}^d} K(x-y) f(y) dy$$

where the function $K(\cdot)$, called the *kernel*, is a function $K : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{C}$ satisfying

1. *decay estimate*: $|K(x)| \leq C_1 \frac{1}{|x|^d}$ for all $x \in \mathbb{R}^d \setminus \{0\}$.
2. *smoothness estimate*: $\int_{|x| > 2|y|} |K(x) - K(x-y)| dx \leq C_2$ for all $y \neq 0$.
3. *cancelation property*: $\int_{r < |x| < s} K(x) dx = 0$ for all $0 < r < s < \infty$.

Proposition 2. The Hilbert transform maps L^p into L^p for any $1 < p < \infty$, and L^1 into $L^{1,\infty}$:

$$\|Hf\|_p \lesssim \|f\|_p, \quad \text{and} \quad \|Hf\|_{1,\infty} \lesssim \|f\|_1.$$

This can still be generalized to *Calderón-Zygmund* operators, which are of the form

$$T(f)(x) := \int_{\mathbb{R}} K(x,y) f(y) dy$$

where the *kernel* satisfies the condition

$$\int_{|x-y|>c|\delta|} |K(x, y) - K(x, \bar{y})| dx \leq A, \text{ for any } \bar{y} \in B(y, \delta).$$

Using a Calderòn-Zygmund decomposition and via interpolation, one can prove the following result:

Theorem 8. *If T is an operator satisfying the conditions above, and if $T : L^q \rightarrow L^q$ for some $1 < q < \infty$, then $T : L^p \rightarrow L^p$ for every $1 < p < q$. Moreover, $T : L^1 \rightarrow L^{1,\infty}$.*

2.1.4 A Unifying Approach: Vector-Valued Convolution Operators

In [2], Benedek, Calderón, and Panzone study vector-valued convolution operators, unifying the theory of singular integrals, maximal operators and that of square functions. To be more precise, we are dealing with the maximal operator

$$\mathcal{M}_\Phi f(x) := \sup_{k \in \mathbb{Z}} |f * \Phi_k(x)|, \quad \text{where } \int_{\mathbb{R}} \Phi dx = 1,$$

and the square function

$$\mathcal{S}f(x) := \left(\sum_{k \in \mathbb{Z}} |f * \psi_k|^2 \right)^{1/2}, \quad \text{where } \text{supp } \hat{\psi} \subseteq \left[-2, -\frac{1}{2} \right] \cup \left[\frac{1}{2}, 2 \right].$$

To begin with, neither the maximal operator nor the square function are linear operators. But if we regard the maximal operator as a map from L^p to $L^p(l^\infty)$, and the square function as a map from L^p to $L^p(l^2)$, they become linear operators. We have the following:

1. *singular integral operator:* $T : L^p \rightarrow L^p, f \mapsto f * K(x)$.
2. *maximal operator:* $M_\Phi : L^p \rightarrow L^p(l^\infty), f \mapsto \{f * \Phi_k(x)\}_k$.

3. *square function*: $\mathcal{S} : L^p \rightarrow L^p(l^2)$, $f \mapsto \{f * \psi_k(x)\}_k$.

This approach was generalized even more by Rubio de Francia, Ruiz, and Torrea in [13], where a Calderón-Zygmund theory for operator-valued kernels is developed. The methods of the proof are similar to the scalar case; the Calderón-Zygmund decomposition can be used in this context as well; the only requirement is that the operator is bounded on some L^r spaces.

Theorem 9 (Thm. 1.1 of [5]). *Let E and F denote Banach spaces, and $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n, x = y\}$. $k \in L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta, \mathcal{L}(E, F))$ is an operator-valued kernel satisfying*

$$\int_{|y'-x| \geq 2|y'-y|} \|k(x, y) - k(x, y')\|_{\mathcal{L}(E, F)} dx + \int_{|y-x'| \geq 2|x-x'|} \|k(x, y) - k(x', y)\|_{\mathcal{L}(E, F)} dy \leq C.$$

Let T be a linear bounded operator from $L^r(\mathbb{R}^n, E)$ into $L^r(\mathbb{R}^n, F)$, for some $1 < r < \infty$.

Then, for all p with $1 < p < \infty$, we have

$$\|Tf\|_{L^p(\mathbb{R}^n, F)} \leq C\|f\|_{L^p(\mathbb{R}^n, E)}.$$

This is a very general setting for singular integral operators. We also note the following result, which is obtained by induction from the previous theorem:

Corollary 1 (Corollary 1.5 of [13]). *Every singular integral operator is bounded from the mixed norm space $L^P(\mathbb{R}^n, E)$ to $L^P(\mathbb{R}^n, F)$, where $P = (p_1, \dots, p_n)$, and $1 < p_i < \infty$.*

Here we emphasize the example of the square function; let $\mathcal{S}_k : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n, l^2)$ be the square function in the k -th component:

$$\{\mathcal{S}_k f(x_1, \dots, x_n)\}_m := \int_{\mathbb{R}} f(x_1, \dots, x_{k-1}, x_k - t, x_{k+1}, \dots, x_n) \psi_m(t) dt.$$

Its adjoint operator $\mathcal{S}_k^* : L^p(\mathbb{R}^n, l^2) \rightarrow L^p(\mathbb{R}^n)$ and is defined by

$$\mathcal{S}_k^* (\{f_m\}) (x_1, \dots, x_n) := \sum_m f_m(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_n) * \psi_m(x_k).$$

Remark 4. Following the Corollary 1 above, we note that both \mathcal{S}_k and \mathcal{S}_k^* are bounded operators on mixed norm spaces, from $L^P(\mathbb{R}^n)$ to $L^P(\mathbb{R}^n, l^2)$ and from $L^P(\mathbb{R}^n, l^2)$ to $L^P(\mathbb{R}^n)$, respectively. It is essential here that $1 < p_j < \infty$, for any $1 \leq j \leq n$. Even more,

$$C_P^{-1} \|f\|_{L_{x_1}^{p_1} \dots L_{x_n}^{p_n}} \leq \|\mathcal{S}_k f\|_{L_{x_1}^{p_1} \dots L_{x_n}^{p_n}} \leq C_P \|f\|_{L_{x_1}^{p_1} \dots L_{x_n}^{p_n}}.$$

We would like to have a similar result for operators that are a mix of maximal operators and square functions. For this, we need a generalization of theorem 9, which is due to D. Fernandez [5]:

Theorem 10 (Thm. 1.2 of [5]). *Let E, F and G be Banach spaces and consider the operator-valued kernels $k_1 \in L_{loc}^1(\mathbb{R}^m \times \mathbb{R}^m \setminus \Delta, \mathcal{L}(E, F))$, and $k_2 \in L_{loc}^1(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta, \mathcal{L}(F, G))$, which satisfy*

$$\int_{|y'-x| \geq 2|y'-y|} \|k(x, y) - k(x, y')\|_{\mathcal{L}_j} dx + \int_{|y-x'| \geq 2|x-x'|} \|k(x, y) - k(x', y)\|_{\mathcal{L}_j} dy \leq C_j.$$

Here $\mathcal{L}_1 = \mathcal{L}(E, F)$ and $\mathcal{L}_2 = \mathcal{L}(F, G)$. Let T_1 and T be linear bounded operators from $L^p(\mathbb{R}^m, E)$ into $L^p(\mathbb{R}^m, F)$, and from $L^p(\mathbb{R}^n \times \mathbb{R}^n, E)$ into $L^p(\mathbb{R}^m \times \mathbb{R}^n, G)$, for all p with $1 < p < \infty$, respectively. Suppose also that T_1 and T satisfy

$$T_1 f(x) = \int_{\mathbb{R}^m} k_1(x, u) f(u) du, \quad \text{and} \quad T f(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} k_2(y, v) k_1(x, u) f(u, v) du dv.$$

Then, for all $P = (p_1, p_2)$ with $1 < p_1, p_2 < \infty$, the linear operator T can be extended to all $L^P(\mathbb{R}^m \times \mathbb{R}^n, E)$ into $L^P(\mathbb{R}^m \times \mathbb{R}^n, G)$ such that

$$\|T f\|_{L^P(\mathbb{R}^m \times \mathbb{R}^n, G)} \leq C \|f\|_{L^P(\mathbb{R}^m \times \mathbb{R}^n, E)}.$$

A careful, inductive application of theorems 9, and 10, yields the following result about mixed square functions and maximal operators:

Theorem 11. (a) *The square function in the y variable, which we denote \mathcal{S}^2 , is a bounded operator on $L_x^{p_1} L_y^{p_2}$ whenever $1 < p_1, p_2 < \infty$:*

$$\left\| \left(\sum_k |P_k^y f(x)|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2}} \lesssim \|f\|_{L_x^{p_1} L_y^{p_2}}.$$

(b) The same is true about the maximal operator in the second variable (denoted \mathcal{M}^2) associated with an approximation of identity:

$$\left\| \sup_k |Q_k^y f(x)| \right\|_{L_x^{p_1} L_y^{p_2}} \lesssim \|f\|_{L_x^{p_1} L_y^{p_2}}.$$

(c) Let $J_S \subseteq \{1, \dots, M\}$ and $J_M \subseteq \{1, \dots, M\}$ be two disjoint subsets of indices. We consider the operator $P_{J_S} Q_{J_M} f$ given by Littlewood-Paley projections in every component $j_s \in J_S$ and maximal operator in every component $j_m \in J_M$; this is a bounded operator on $L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}$, provided $1 < p_1, \dots, p_M < \infty$:

$$\|S^{J_S} \mathcal{M}^{J_M} f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}} \lesssim \|f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}}.$$

(d) Moreover, if there are only square functions appearing, we have an equivalence of norms:

$$\|S^{J_S} f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}} \simeq \|f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}}.$$

2.2 Interpolation

Interpolation theory originates with the works of Riesz and Thorin (in the case of complex interpolation), and Marcinkiewicz for real interpolation methods. We will mention a theorem of Marcinkiewicz, for this approach inspired the interpolation theory in the multilinear setting.

Theorem 12 (Marcinkiewicz Interpolation Theorem). *Let $0 < p_0 < p < p_1 \leq \infty$, and let T be a (sub-)linear operator defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$, and taking values in the space of measurable functions on \mathbb{R}^n . Assume that T is bounded from L^{p_0} to $L^{p_0, \infty}$, and from L^{p_1} to $L^{p_1, \infty}$:*

$$\|T(f)\|_{p_0, \infty} \leq C_0 \|f\|_{p_0} \quad \text{for all } f \in L^{p_0}(\mathbb{R}^n), \quad \text{and}$$

$$\|T(f)\|_{p_1, \infty} \leq C_1 \|f\|_{p_1} \quad \text{for all } f \in L^{p_1}(\mathbb{R}^n).$$

Then for any $p_0 < p < p_1$, T is a bounded operator from L^p to L^p :

$$\|T(f)\|_{p, \infty} \leq C \|f\|_p, \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

Now we will focus on the multi-linear case, though our applications will deal with bilinear operators and hence, trilinear forms. The first results on multilinear interpolation are due to Strichartz [29].

Definition 6. For a subset $E \subset \mathbb{R}$ of finite measure, define

$$X(E) = \{f : |f| \leq \mathbf{1}_E \text{ a.e.}\}.$$

V will denote the linear span of all $X(E)$.

Definition 7. A tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ is called admissible if for all $1 \leq i \leq n$

$$-\infty < \alpha_i < 1, \quad \alpha_1 + \dots + \alpha_n = 1,$$

and there is at most one index j_0 so that $\alpha_{j_0} < 0$. We call an index good if $\alpha_i > 0$ and bad if $\alpha_i \leq 0$.

Definition 8. A multilinear form $\Lambda : V \times \dots \times V \rightarrow \mathbb{C}$ is of restricted type $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 \leq \alpha_i \leq 1$ if there exists a constant C (possibly depending on α) such that for each tuple $E = (E_1, \dots, E_n)$ of measurable subsets of \mathbb{R} and for each tuple $f = (f_1, \dots, f_n)$ with $f_j \in X(E_j)$, we have

$$|\Lambda(f_1, \dots, f_n)| \leq C \prod_j |E_j|^{\alpha_j}.$$

The condition $\sum_j \alpha_j = 1$ comes from scaling invariance of multilinear forms. In the case of trilinear forms, the admissible pairs can be described by a triangle in the hyperplane $\alpha_1 + \alpha_2 + \alpha_3 = 1$:

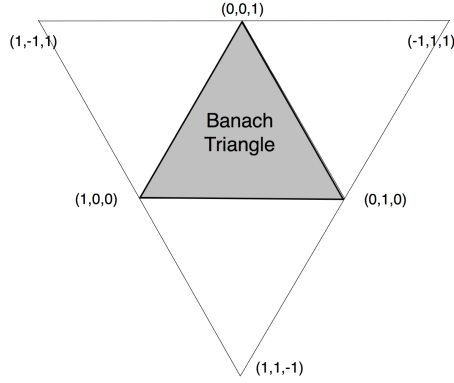


Figure 2.1: Set of admissible pairs

We note that the *Banach triangle* correspond to “good tuples”, and consequently every $\alpha_j = \frac{1}{p_j}$, where L^{p_j} is a Banach space (and so $1 < p_j < \infty$).

Theorem 13 (Similar to Theorem 3.2 in [30]). *Let $\beta = (\beta_1, \dots, \beta_n)$ be a tuple of real numbers such that $\sum_j \beta_j = 1$ and $\beta_j > 0$ for all j . Assume Λ is of restricted type α for all α in a neighborhood of β satisfying $\sum_j \alpha_j = 1$, with constant $C(\alpha)$ depending continuously on α . Then Λ is of strong type β with constant $C(\beta)$:*

$$|\Lambda(f_1, \dots, f_n)| \leq C(\beta) \prod_{j=1}^n \|f_j\|_{1/\beta_j} \quad \text{for all } f_j \in V.$$

Definition 9. *Let α be an n -tuple of real numbers and assume $\alpha_j \leq 1$ for all j . An n -linear form Λ is called of generalized restricted type α if there is a constant C (possibly depending on α) such that for all tuples $E = (E_1, \dots, E_n)$, there is an index j_0 and a major subset $E'_{j_0} \subseteq E_{j_0}$ so that for all tuples $f = (f_1, \dots, f_n)$ with $f_j \in X(E_j)$ for $j \neq j_0$ and $f_{j_0} \in X(E'_{j_0})$*

$$|\Lambda(f_1, \dots, f_n)| \leq C \prod_{j=1}^n |E_j|^{\alpha_j}.$$

In some of our applications, the constants appearing in the interpolation theorems (which do not depend on the functions, but could depend in the tuple

$\alpha = (\alpha_1, \dots, \alpha_n)$ are going to play an important role. For this reason, here we state interpolation results from [30], with a slightly modified statement that keeps track of the constants.

Proposition 3 (Similar to Lemma 3.6 in [30]). *If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a good tuple, and Λ is of generalized restricted type α with constant $C(\alpha)$ and the major subset corresponds to the index j_0 , then Λ is of restricted type α with constant $\frac{C(\alpha)}{1 - 2^{-j_0}}$.*

Theorem 14. (Thm. 3.8 of [30]) *Assume Λ is of generalized restricted type β where $\sum_j \beta_j = 1$. Assume $\beta_k > 0$ for $1 \leq k \leq n-1$ and $\beta_n \leq 0$. Assume Λ is also of generalized restricted type α with constant $C(\alpha)$ (continuously depending on α) for all α in a neighborhood of β satisfying $\sum_j \alpha_j = 1$. Then the dual form (in the n -th function) T satisfies*

$$\|T(f_1, \dots, f_{n-1})\|_{1/(1-\beta_n)} \leq C(\beta) \prod_{j=1}^{n-1} \|f_j\|_{1/\beta_j}. \quad (2.7)$$

Interpolation for Banach-valued Functions

Let X be a Banach space; we want to define $L^p(\mathbb{R}; X)$ spaces and adapt interpolation results for vector-valued operators.

We say that $F \in L^p(\mathbb{R}; X)$ provided

$$\|f\|_{L^p(\mathbb{R}; X)} := \left(\int_{\mathbb{R}} \|F(x)\|_X^p dx \right)^{1/p} < \infty.$$

The question of integrability of $F(x)$ is reduced to the Lebesgue integrability of $\|F(x)\|_X$. A more rigorous definition of Banach-valued L^p spaces would lead to the concept of *Bochner integral*. To make some sense of vector-valued functions however, we say that a function $F : \mathbb{R} \rightarrow X$ is *weakly measurable* if $x \mapsto \langle u^*, F(x) \rangle$ is measurable as a function of x for any $u^* \in X^*$.

The vector spaces appearing in this thesis are $L'(\mathbb{R})$, l' , or iterations of these, and the abstract approach can be avoided.

Proposition 4. *The set of X -valued functions of the form*

$$\left\{ \sum_{j=1}^N \mathbf{1}_{E_j} u_j : E_j \text{ are subsets of } \mathbb{R} \text{ of finite measure, } u_j \in X \right\}$$

is dense in $L^p(\mathbb{R}; X)$ whenever $0 < p < \infty$.

In the scalar case, a very important role in interpolation theory is played by the set of functions which are bounded above by the characteristic function of a set of finite measure, that is $|f(x)| \leq \mathbf{1}_E(x)$. In the vector-valued case, this collection of functions is replaced by

$$\{f : \|f(x)\|_X \leq \mathbf{1}_E \text{ a.e.}\}.$$

V_X will denote the linear span of the above sets, for all measurable $E \subseteq \mathbb{R}$ of finite measure. The role that V_X is playing here is that of a dense subset of $L^p(\mathbb{R}; X)$ spaces. We prove results about functions in V_X first(i.e. inequalities); then we extend these results to L^p spaces, but the closure will depend on the norm that we are using. In some sense, V_X allows us to disregard the domain of a given operator before we prove its boundedness.

Proposition 5 (Dualization of the Norm). *For any $F \in L^p(\mathbb{R}; X)$ and any $1 \leq p < \infty$, we have*

$$\|F\|_{L^p(\mathbb{R}; X)} := \sup_{\|G\|_{L^{p'}(\mathbb{R}; X^*)} \leq 1} \left| \int_{\mathbb{R}} \langle G(x), F(x) \rangle dx \right|.$$

Multilinear Interpolation

We are interested in interpolation results for vector-valued multi-linear operators (or multi-sublinear) of the form

$$\vec{T} : L^{p_1}(\mathbb{R}^n; X_1) \times \dots \times L^{p_{m-1}}(\mathbb{R}^n; X_{m-1}) \rightarrow L^{p_m}(\mathbb{R}^n; X_m).$$

There is a multilinear form associated with the operator $\Lambda : (V_{X_1}) \times \dots \times (V_{X_m}) \rightarrow \mathbb{C}$:

$$\Lambda(F_1, \dots, F_{m-1}, F_m) = \int_{\mathbb{R}^n} \langle T(F_1, \dots, F_{m-1})(x), F_m(x) \rangle dx.$$

Here $\langle \cdot, \cdot \rangle$ denote the $\langle x, x^* \rangle$ pairing, where $x \in X$ and $x^* \in X^*$. The definitions and proofs from the scalar case are adaptable to the vector-valued situation. We present these here, adapting the equivalent statements from [30].

Definition 10. A tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ is called *admissible* if $\alpha_1 + \dots + \alpha_n = 1$, $\alpha_1, \dots, \alpha_n < 1$ and there exists at most one index j_0 for which we have $\alpha_{j_0} < 0$.

A multi-sublinear form Λ as above is of *restricted type* $\alpha = (\alpha_1, \dots, \alpha_n)$ for a good admissible tuple α if there exists a constant C so that for each tuple $E = (E_1, \dots, E_n)$ of measurable subsets of \mathbb{R} , and for each tuple $f = (f_1, \dots, f_n)$ with $\|f_j\|_X \leq \mathbf{1}_{E_j}$, we have

$$|\Lambda(f_1, \dots, f_n)| \leq C |E_1|^{\alpha_1} \cdot \dots \cdot |E_n|^{\alpha_n}.$$

Proposition 6 (Equivalent of Thm. 3.2 of [30]). Let $\beta = (\beta_1, \dots, \beta_n)$ be an admissible tuple of real numbers such that $\beta_j > 0$ for all j . Assume that Λ is of restricted type α for all admissible tuples α in a neighborhood of β . Then there is a constant C such that for all $f_j \in V_{X_j}$,

$$|\Lambda(f_1, \dots, f_n)| \leq C \|f_1\|_{L^{1/\beta_1}(\mathbb{R}; X_1)} \cdot \dots \cdot \|f_n\|_{L^{1/\beta_n}(\mathbb{R}; X_n)}.$$

Proof. Using re-arrangements, we can assume that the f_j s have the property that $\|f_j\|_{X_j}$ are non-increasing, supported on $(0, \infty)$. Then for every $1 \leq j \leq n$, we have

the splitting

$$f_j = \sum_{k_j} f_j \mathbf{1}_{[2^{k_j}, 2^{k_j+1})}, \quad \text{with } \|f_j(x) \mathbf{1}_{[2^{k_j}, 2^{k_j+1})}(x)\|_{X_j} \leq \|f_j(2^{k_j})\|_{X_j} \mathbf{1}_{[2^{k_j}, 2^{k_j+1})}(x).$$

The restricted type estimates and the multi-sublinearity imply that

$$|\Lambda(f_1, \dots, f_n)| \leq C \sum_{k_1, \dots, k_n} \prod_j \|f_j(2^{k_j})\|_{X_j} 2^{\alpha_j k_j}.$$

From here on the proof is identical to the scalar case: the tuple α depends on k_1, \dots, k_n , and is chosen from a neighborhood of β so that

$$\sum_j (\beta_j - \alpha_j)(k_j - k) \geq \epsilon \max_j |k_j - k|,$$

where $k = \frac{k_1 + \dots + k_n}{n}$ is the average of the k_j s. This implies

$$\begin{aligned} |\Lambda(f_1, \dots, f_n)| &\leq C \sum_{k_1, \dots, k_n} 2^{-\epsilon(\max_j |k_j - k|)} \prod_j \|f_j(2^{k_j})\|_{X_j} 2^{\beta_j k_j} = \\ &= C \sum_{k_1 - k, \dots, k_{n-1} - k} 2^{-\epsilon(\max_j |k_j - k|)} \sum_k \prod_j \|f_j(2^{k_j - k + k})\|_{X_j} 2^{\beta_j(k_j - k + k)} \leq \\ &\leq C \sum_{k_1 - k, \dots, k_{n-1} - k} 2^{-\epsilon(\max_j |k_j - k|)} \prod_j \|f_j\|_{L^{1/\beta_j}(\mathbb{R}; X_j)} \leq C \prod_j \|f_j\|_{L^{1/\beta_j}(\mathbb{R}; X_j)} \end{aligned}$$

□

So in order to prove the boundedness of an operator on some $L^p(\mathbb{R}; X)$ spaces, it suffices to check restricted type estimates. For multilinear operators, it often happens that the target space is an L^p space with $p < 1$, which is only a quasi-Banach space. There is a way of dualizing this particular class of quasi-Banach norms, by removing small exceptional sets. With this idea in mind, we introduce the following definition and results:

Definition 11. Let α be an admissible tuple; the n -sublinear form Λ is of generalized restricted type α if there is a constant C such that for all tuples $E = (E_1, \dots, E_n)$ there

is an index j_0 and a major subset E'_{j_0} of E_{j_0} (that is, $|E'_{j_0}| \geq |E_{j_0}|/2$) such that for all tuples $f = (f_1, \dots, f_n)$ with $\|f_j\|_{X_j} \leq \mathbf{1}_{E_j}$ for $j \neq j_0$, and $\|f_{j_0}\|_{X_{j_0}} \leq \mathbf{1}_{E'_{j_0}}$, we have

$$|\Lambda(f_1, \dots, f_n)| \leq C \prod_j |E_j|^{\alpha_j}.$$

Proposition 7. *If Λ is of generalized restricted type $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\alpha_j > 0$ for all j , then Λ is of restricted type α .*

On the other hand, if one of the indices α_j is ≤ 0 , the generalized restricted type implies weak- L^p estimates. This applies to the case when the multi-sublinear form is given by

$$\Lambda(f_1, \dots, f_n) = \int_{\mathbb{R}} \langle T(f_1, \dots, f_{n-1})(x), f_n(x) \rangle dx \quad (2.8)$$

and corresponds to an operator T defined on $V_{X_1} \times \dots \times V_{X_{n-1}}$ and taking values in X_n^* .

Proposition 8. *Let Λ be a multi-sublinear form as in 2.8, and $\alpha = (\alpha_1, \dots, \alpha_n)$ an admissible tuple with $\alpha_n \leq 0$. Assuming that Λ is of generalized restricted type α , we have*

$$\lambda |\{x : \|T(f_1, \dots, f_{n-1})(x)\|_{X_n} > \lambda\}|^{1/1-\alpha_n} \leq A \prod_{j=1}^{n-1} |E_j|^{\alpha_j}$$

for all tuples $f = (f_1, \dots, f_{n-1})$ with $\|f_j\|_{X_j} \leq \mathbf{1}_{E_j}$.

Proposition 9. *Assume Λ is of generalized restricted type β where β is an admissible tuple with $\beta_n \leq 0$. Assume Λ is also of generalized restricted type α for all admissible tuples α in a neighborhood of β . Then T satisfies*

$$\|T(f_1, \dots, f_{n-1})\|_{L^{1/1-\beta_n}(\mathbb{R}; X_n)} \leq C \prod_{j=1}^{n-1} \|f_j\|_{L^{1/\beta_j}(\mathbb{R}; X_j)}. \quad (2.9)$$

The proofs of the last two propositions follow exactly the same ideas as the scalar case, with minor differences.

2.3 Time-Frequency Analysis: Paraproducts and *BHT*

While here we use Chapter 6 of [6] as a black box, we recall a few definitions and results to ease the reading of the presentation. Essential are the notions of *size* and *energy*, which are averages over certain subsets of phase-frequency space. The techniques employed for proving vector-valued estimates for *BHT* and paraproducts are of such nature that the proof of the scalar case is carried out through the process. For this reason, we can say that the proof of the boundedness of these operators is concealed in chapters 4 and 5. What is left out are the very important Propositions 11 and 10.

We briefly mention the system of *Haar functions*, because they will appear later on, in chapters 3 and 7.

Definition 12. *Given a dyadic interval, we denote by I_{left} its left half, and by I_{right} its right half. Then*

$$h_I(x) := \frac{1}{|I|^{1/2}} (\mathbf{1}_{I_{left}} - \mathbf{1}_{I_{right}}).$$

Then the Haar system $\{h_I\}_{I \in \mathcal{D}}$ forms an orthonormal basis in $L^2(\mathbb{R})$, and is an unconditional basis in $L^p(\mathbb{R})$ for any p so that $1 < p < \infty$.

Notation 2. *For any interval $I \subset \mathbb{R}$, define*

$$\tilde{\chi}_I(x) := \left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{-100}.$$

Definition 13. *Given an interval I and a function ϕ , we say that ϕ is adapted to I if*

$$|\phi^{(l)}(x)| \leq C_l C_M \frac{1}{|I|^l} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^M}$$

for sufficiently many derivatives l , and a large number M .

Definition 14. A tile is a rectangle $P = I_P \times \omega_P$ with the property that $I_P, \omega_P \in \mathcal{D}$ or ω_P is in a shifted variant of \mathcal{D} . Abusing notation, we define a tri-tile to be a tuple $P = (P_1, P_2, P_3)$ where each P_i is a tile as defined above and the spatial intervals are the same: $I_{P_i} = I_P$ for all $1 \leq i \leq 3$.

Definition 15 (Order relation). Given two tiles P and P' , we say $P' < P$ if $I_{P'} \subsetneq I_P$ and $\omega_P \subset 3\omega_{P'}$. $P' \leq P$ if $P' < P$ or $P' = P$. Also, $P' \lesssim P$ if $I_{P'} \subset I_P$ and $\omega_P \subseteq 100\omega_{P'}$, and $P' \lesssim' P$ if $P' \lesssim P$ but $P' \not\leq P$.

Definition 16. A collection \mathbb{P} of tri-tiles is said to have rank 1 if for any $P, P' \in \mathbb{P}$:

- if the tri-tiles are distinct $P \neq P'$, then $P'_j \neq P_j$ for all $1 \leq j \leq 3$.
- if $\omega_{P_{j_0}} = \omega_{P'_{j_0}}$ for some j_0 , then $\omega_{P_j} = \omega_{P'_j}$ for all $1 \leq j \leq 3$.
- if $P'_{j_0} \leq P_{j_0}$ for some j_0 , then $P'_j \lesssim P_j$ for all $1 \leq j \leq 3$.
- if in addition to $P'_{j_0} \leq P_{j_0}$ one also assumes $|I_{P'}| \ll |I_P|$, then $P'_j \lesssim' P_j$ for all $j \neq j_0$.

Definition 17. Let \mathbb{P} be a sparse rank 1 collection of tri-tiles, and let $1 \leq j \leq 3$. A subcollection T of \mathbb{P} is called a j -tree if and only if there exists a tri-tile P_T (called the top of the tree) such that $P_j \leq P_{T,j}$ for all $P \in T$. We write I_T for I_{P_T} and $\omega_{T,j}$ for $\omega_{P_T,j}$ and we say T is a tree if it is a j -tree for some $1 \leq j \leq 3$.

Definition 18. Let $1 \leq i \leq 3$. A finite sequence of trees T_1, \dots, T_M is said to be a chain of strongly i -disjoint trees if and only if

- (i) $P_i \neq P'_i$ for every $P \in T_{l_1}$ and $P' \in T_{l_2}$, with $l_1 \neq l_2$;
- (ii) whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 \neq l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$, then if $|\omega_{P_i}| < |\omega_{P'_i}|$ one has $I_{P'} \cap I_{T_{l_1}} = \emptyset$, and if $|\omega_{P'_i}| < |\omega_{P_i}|$, one has $I_P \cap I_{T_{l_2}} = \emptyset$.

(iii) whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 < l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$ and $|\omega_{P_i}| = |\omega_{P'_i}|$, then $I_{P'} \cap I_{T_{l_1}} = \emptyset$.

Definition 19. Let P be a tile. A wave packet on P is a smooth function ϕ_P which has Fourier support inside $\frac{9}{10}\omega_P$ and is L^2 -adapted to I_P in the sense that

$$|\phi_P^{(l)}(x)| \leq C_{l,M} \frac{1}{|I_P|^{1/2+l}} \left(1 + \frac{\text{dist}(x, I_P)}{|I_P|}\right)^{-M} \quad (2.10)$$

for sufficiently many derivatives l , and any $M > 0$.

2.4 Singular Bilinear Operators

There are two main bilinear operators that we are studying in this thesis. These are

(i) *paraproducts*:

$$\Pi(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t)g(x-s)k(s, t)dsdt,$$

where $k(s, t)$ is the two-dimensional equivalent of the *kernel* of a singular integral operator.

(ii) the *bilinear Hilbert transform*, *BHT* in short:

$$BHT(f, g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)\frac{1}{t}dt.$$

Equivalently, we have the Fourier representations:

(i) for paraproducts:

$$\Pi(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi_1)\hat{g}(\xi_2)m(\xi_1, \xi_2)e^{2\pi i x(\xi_1 + \xi_2)}d\xi_1d\xi_2,$$

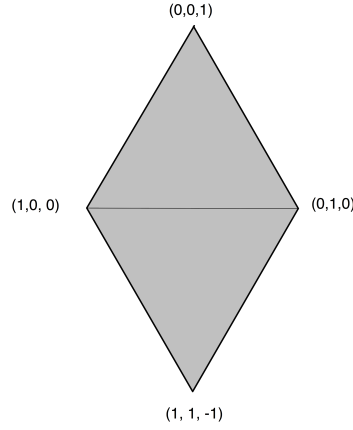


Figure 2.2: Range of paraproducts Π

where $m(\xi_1, \xi_2)$ is a two-dimensional *Marcinkiewicz-Mikhlin-Hörmander* multiplier, smooth away from $(0, 0)$, and satisfying $|\partial^\alpha m(\xi)| \lesssim |\xi|^{-\alpha}$.

(ii) for *BHT*:

$$BHT(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi_1) \hat{g}(\xi_2) \operatorname{sgn}(\xi_1 - \xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2.$$

The range of these operators is also quite different.

Theorem 15 (Coifman, Meyer). *The paraproduct Π , associated to a Marcinkiewicz-Mikhlin-Hörmander multiplier m , maps $L^p \times L^q$ into L^s provided that $1 < p, q \leq \infty$, $\frac{1}{2} < s < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.*

Theorem 16 (Lacey, Thiele, [7]). *BHT maps $L^p \times L^q$ into L^s provided that $1 < p, q \leq \infty$, $\frac{2}{3} < s < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.*

2.4.1 Paraproducts

We are mainly interested in a special case of paraproducts, that arise as mollifiers of products of two functions $f \cdot g$. Using Littlewood-Paley decompositions

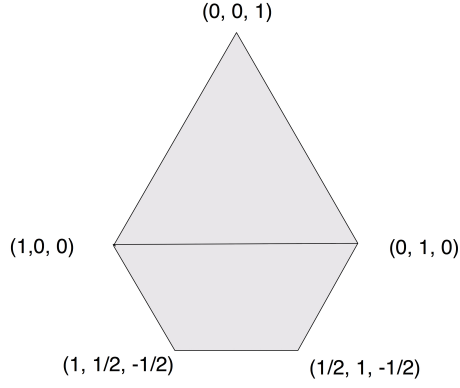


Figure 2.3: Range for *BHT* operator

for f and g respectively, we have

$$\begin{aligned}
f \cdot g &:= \left(\sum_k f * \psi_k \right) \left(\sum_l g * \psi_l \right) = \sum_{k,l} (f * \psi_k) \cdot (g * \psi_l) = \\
&= \sum_{k \sim l} (f * \psi_k) \cdot (g * \psi_l) + \sum_{k \ll l} (f * \psi_k) \cdot (g * \psi_l) + \sum_{l \ll k} (f * \psi_k) \cdot (g * \psi_l) = \\
&= \underbrace{\sum_k (f * \psi_k) \cdot (g * \psi_k)}_{(I)} + \underbrace{\sum_k (f * \varphi_k) \cdot (g * \psi_k)}_{(II)} + \underbrace{\sum_k (f * \psi_k) \cdot (g * \varphi_k)}_{(III)}
\end{aligned}$$

For (I), $\widehat{f * \psi_k}$ is supported in $|\xi| \sim 2^k$, $\widehat{g * \psi_k}$ is supported in $|\xi| \sim 2^k$, and hence their convolution will be supported in $|\xi| \leq 2^k$. So we have

$$\sum_k (f * \psi_k) \cdot (g * \psi_k)(x) = \sum_k ((f * \psi_k) \cdot (g * \psi_k)) * \varphi_k(x).$$

In this way, $(I) = \sum_k Q_k(P_k f \cdot P_k g)$. Similarly,

$$(II) = \sum_k ((f * \varphi_k) \cdot (g * \psi_k)) * \psi_k(x) = \sum_k P_k(Q_k f \cdot P_k g).$$

$$(III) = \sum_k ((f * \psi_k) \cdot (g * \varphi_k)) * \psi_k(x) = \sum_k P_k(P_k f \cdot Q_k g).$$

A paraproduct Π will represent any of the expressions (I), (II) or (III). We

will also need to understand a slightly different type of operator,

$$\tilde{\Pi}^\alpha(f, g)(x) := \sum_k ((f * \psi_k) \cdot (g * \psi_k)) * \varphi_k^\alpha(x),$$

where the exterior functions $\varphi_k^\alpha = 2^k \varphi^\alpha(2^k x)$ have a slower decay at infinity:

$$|\varphi^\alpha(x)| \lesssim \frac{1}{(1 + |x|)^{1+\alpha}}.$$

We refer to the expression $\tilde{\Pi}^\alpha$ as a *generalized paraproduct*.

Definition 20. A discretized paraproduct associated to a family of intervals $\{I\}_{I \in \mathcal{J}}$ is defined by

$$\Pi(f, g)(x) = \sum_{I \in \mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1 \rangle \langle g, \phi_I^2 \rangle \phi_I^3(x),$$

where two of the families $(\phi_I^j)_{I \in \mathcal{I}}$ are lacunary, and the third one is non-lacunary.

We still need to recall a few definitions and results, which are cited from [6].

The concepts of *sizes* and *energies* play an essential role in the methods of the proofs that we pursue.

Definition 21. Let \mathcal{J} be a family of dyadic intervals. For any $1 \leq j \leq 3$, we define

$$\begin{aligned} \text{size}_{\mathcal{J}}(\langle f, \phi_I^j \rangle_{I \in \mathcal{J}}) &= \sup_{I \in \mathcal{J}} \frac{|\langle f, \phi_I^j \rangle|}{|I|^{1/2}}, \quad \text{if } (\phi_I^j)_I \text{ is non-lacunary and} \\ \text{size}_{\mathcal{J}}(\langle f, \phi_I^j \rangle_{I \in \mathcal{J}}) &= \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I \in \mathcal{J}}} \frac{|\langle f, \phi_I^j \rangle|}{|I|^{1/2}} \mathbf{1}_I \right)^{1/2} \right\|_{1, \infty}, \quad \text{if } (\phi_I^j)_I \text{ is lacunary.} \end{aligned}$$

The energy is defined as

$$\text{energy}_{\mathcal{J}}^j(\langle f, \phi_I^j \rangle_{I \in \mathcal{J}}) := \sup_{n \in \mathbb{Z}} \sup_{\mathbb{D}} \left(\sum_{I \in \mathbb{D}} |I| \right)$$

where \mathbb{D} ranges over all disjoint collections of intervals I_0 with the property that

$$\frac{|\langle f, \phi_{I_0}^j \rangle|}{|I_0|^{1/2}} \geq 2^n \text{ or } \frac{1}{|I_0|^{1/2}} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I \in \mathcal{J}}} \frac{|\langle f, \phi_I^j \rangle|}{|I|^{1/2}} \mathbf{1}_I \right)^{1/2} \right\|_{1, \infty} \geq 2^n.$$

Lemma 1 (Lemma 2.13 of [6]). *If F is an L^1 function and $1 \leq j \leq 3$, then*

$$\text{size}_j^j(\langle F, \phi_I^j \rangle_{I \in \mathcal{J}}) \lesssim \sup_{I \in \mathcal{J}} \frac{1}{|I|} \int_{\mathbb{R}} |F| \tilde{\chi}_I^M dx$$

for $M > 0$, with implicit constants depending on M .

Lemma 2. (Lemma 2.14 of [6]) *If F is an L^1 function and $1 \leq j \leq 3$, then*

$$\text{energy}_j^j(\langle F, \phi_I^j \rangle_{I \in \mathcal{J}}) \lesssim \|F\|_1.$$

Proposition 10 (Proposition 2.12 of [6]). *Given a paraproduct Π associated with a family \mathcal{J} of intervals,*

$$\begin{aligned} |\Lambda_\Pi(f_1, f_2, f_3)| &= \left| \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f_1, \phi_I^1 \rangle \langle f_2, \phi_I^2 \rangle \langle f_3, \phi_I^3 \rangle \right| \lesssim \\ &\lesssim \prod_{j=1}^3 \left(\text{size}_j^{(j)}(\langle f_j, \phi_I^j \rangle_{I \in \mathcal{J}}) \right)^{1-\theta_j} \left(\text{energy}_j^{(j)}(\langle f_j, \phi_I^j \rangle_{I \in \mathcal{J}}) \right)^{\theta_j} \end{aligned}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ such that $\theta_1 + \theta_2 + \theta_3 = 1$, where the implicit constant depends on $\theta_1, \theta_2, \theta_3$ only.

Remark 5. *The sizes are discrete BMO type of averages, and a very important result of John and Nirenberg states that all the $BMO(p)$ norms are equivalent, i.e.*

$$\sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|^{1/q}} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I \in \mathcal{J}}} \frac{|\langle f, \phi_I^j \rangle|}{|I|^{1/2}} \mathbf{1}_I \right)^{1/2} \right\|_q \simeq \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I \in \mathcal{J}}} \frac{|\langle f, \phi_I^j \rangle|}{|I|^{1/2}} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty},$$

for any $1 < q < \infty$.

2.4.2 Bilinear Hilbert Transform

For the *BHT* operator, we can use a Whitney decomposition of the region $\{\xi_1 < \xi_2\}$, and eventually arrive to the discretized model

$$BHT_{\mathbb{P}}(f, g)(x) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{1/2}} \langle f, \phi_{P_1}^1 \rangle \langle g, \phi_{P_2}^2 \rangle \phi_{P_3}^3(x) \quad (2.11)$$

where the family \mathbb{P} of tri-tiles is sparse and has rank 1, and the $(\phi_{P_j}^j)_{P \in \mathbb{P}}$ are wave packets associated to the tiles P_j .

Associated to this operator we have a tri-linear form

$$\Lambda_{BHT, \mathbb{P}}(f, g, h) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|} \langle f, \phi_{P_1}^1 \rangle \langle g, \phi_{P_2}^2 \rangle \langle h, \phi_{P_3}^3 \rangle. \quad (2.12)$$

The sizes and energies appear again in this case, but they are somehow different.

Definition 22. Let \mathbb{P} be a finite collection of tri-tiles, let $j \in \{1, 2, 3\}$, and let f be an arbitrary function. We define the size of the sequence $\langle f, \Phi_{P_j}^j \rangle_P$ by

$$\text{size} \left(\langle f, \Phi_{P_j}^j \rangle_P \right) := \sup_{T \subseteq \mathbb{P}} \left(\frac{1}{|I_T|} \sum_{P \in T} |\langle f, \phi_{P_j}^j \rangle|^2 \right)^{1/2}, \quad (2.13)$$

where T ranges over all trees in \mathbb{P} that are i -trees for some $i \neq j$.

We will mostly need the simpler "sizes"

$$\widetilde{\text{size}}_{\mathbb{P}}(f) := \sup_{P \in \mathbb{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f| \tilde{\chi}_{I_P}^m dx.$$

Definition 23. Let \mathbb{P} be a finite collection of tri-tiles, $j \in \{1, 2, 3\}$ and let f be a fixed function. We define the energy of the sequence $\langle f, \Phi_{P_j}^j \rangle_P$ by

$$\text{energy} \left(\langle f, \Phi_{P_j}^j \rangle_P \right) := \sup_{n \in \mathbb{Z}} \sup_{\mathbb{T}} 2^n \left(\sum_{T \in \mathbb{T}} |I_T| \right)^{1/2} \quad (2.14)$$

where \mathbb{T} ranges over all chains of strongly j -disjoint trees in \mathbb{P} (which are i -trees for some $i \neq j$) having the property that

$$\left(\sum_{P \in T} |\langle f, \phi_{P_j}^j \rangle|^2 \right)^{1/2} \geq 2^n |I_T|^{1/2}$$

for all $T \in \mathbb{T}$ and such that

$$\left(\sum_{P \in T'} |\langle f, \phi_{P_j}^j \rangle|^2 \right)^{1/2} \leq 2^{n+1} |I_{T'}|^{1/2}$$

for all subtrees $T' \subseteq T \in \mathbb{T}$.

We have the following estimates for the trilinear form, size and energy :

Proposition 11 (Prop. 6.12 of [6]). *Let \mathbb{P} be a finite collection of tri-tiles. Then*

$$\Lambda_{BHT;\mathbb{P}}(f_1, f_2, f_3) \lesssim \prod_{j=1}^3 \left(\text{size} \left(\langle f_j, \phi_{P_j}^j \rangle_P \right) \right)^{\theta_j} \left(\text{energy} \left(\langle f_j, \phi_{P_j}^j \rangle_P \right) \right)^{1-\theta_j}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$; the implicit constants depend on the θ_j but are independent of the other parameters.

Lemma 3 (Lemma 6.13 of [6]). *Let $j \in \{1, 2, 3\}$ and let E be a set of finite measure. Then for every $|f| \leq \chi_E$ one has*

$$\text{size} \left(\langle f, \phi_{P_j}^j \rangle_P \right) \lesssim \sup_{P \in \mathbb{P}} \frac{1}{|I_P|} \int_E \tilde{\chi}_{I_P}^M dx$$

for all $M > 0$, with implicit constants depending on M .

Lemma 4 (Lemma 6.14 of [6]). *Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then*

$$\text{energy} \left(\langle f, \phi_{P_j}^j \rangle_P \right) \lesssim \|f\|_2.$$

Remark 6. *It is worth pointing out a few similarities and differences between para-products and BHT:*

- a) *We can see from the multiplier representations that the symbol m of the paraproduct Π is singular only at $(0, 0)$, while for BHT the symbol $\text{sng}(\cdot)$ is singular along the line $\{\xi_1 = \xi_2\}$.*
- b) *Regarding the discretized versions, $\Pi_{\vec{j}}$ corresponds to a rank 0 family of tiles (by knowing the scale, or the length of the interval I , we know exactly where the tile $P = I \times \omega$ is positioned). In contrast, $BHT_{\mathbb{P}}$ is associated to a rank 1 family of tiles.*
- c) *The restriction of $BHT_{\mathbb{P}}$ to a tree operates as an affine transformation of a para-product.*

d) The energies for BHT are L^2 quantities, while the ones for Π are L^1 . This is why the range of BHT is smaller than that of paraproducts. If we choose to view BHT as a superposition of translated paraproducts, then we need to use their L^2 orthogonality, and we can't expect energies bellow L^2 .

CHAPTER 3

A RUBIO DE FRANCIA THEOREM FOR PARAPRODUCTS

3.1 Maximal Paraproduct

A maximal paraproducts is defined by:

$$\tilde{\Pi}^{\max}(f, g)(x) := \sup_{m \in \mathbb{Z}} \sum_{\substack{I \in \mathcal{I} \\ |I| \geq 2^m}} \frac{\langle f, \varphi_I \rangle}{|I|^{1/2}} \langle g, \psi_I \rangle \psi_I(x). \quad (3.1)$$

This version uses wave packets, but instead we could use Haar functions for the last component.

$$\Pi^{\max}(f, g)(x) := \sup_{m \in \mathbb{Z}} \sum_{\substack{I \in \mathcal{I} \\ |I| \geq 2^m}} \frac{\langle f, \varphi_I \rangle}{|I|^{1/2}} \langle g, \psi_I \rangle h_I(x). \quad (3.2)$$

Even more, there is a variant of maximal paraproducts with shifted Haar functions in the last component:

$$\Pi_{\mathfrak{n}}^{\max}(f, g)(x) := \sup_{m \in \mathbb{Z}} \sum_{\substack{I \in \mathcal{I} \\ |I| \geq 2^m}} \frac{1}{|I|} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle h_{I_{\mathfrak{n}}}(x).$$

In [24], Lacey proves the boundedness of a maximal *BHT* operator, which, for the discretized model, is defined as

$$BHT^{\max}(f, g) := \sup_{k \in \mathbb{Z}} \left| \sum_{P \in \mathbb{P} |I_P| \geq 2^k} \frac{1}{|I_P|^{1/2}} \langle f, \phi_{P_1}^1 \rangle \langle g, \phi_{P_2}^2 \rangle \phi_{P_3}^3(x) \right|.$$

It is proved that $BHT^{\max} : L^p \times L^q \rightarrow L^s$, whenever $\frac{2}{3} < s < \infty$. The case $\frac{2}{3} < s \leq 1$ is much more difficult than the $1 < s < \infty$ case, and it employs a complicated lemma of Bourgain from [4].

The case of interest for us, regarding $\Pi_{\mathfrak{n}}^{\max}$ is $L^2 \times L^2 \rightarrow L^1$, which will be necessary in the following subsection, for dealing with Rubio de Francia operator

associated with paraproducts. We will be focusing on the case $\frac{1}{2} < s \leq 2$, so this range doesn't follow directly from the BHT^{max} theorem. Also, the proof is going to be somehow simpler, and it makes use only of localizations and stopping time arguments.

For a.e. x , we can linearize Π_n^{max} by letting $m(x)$ denote the integer where the sup is attained. In this case, the outside function is a shifted Haar function, and hence $h_{I_n}(x) = \pm \frac{1}{\sqrt{I}} \mathbf{1}_{I_n}$. Since the collection \mathcal{J} is finite, Π_n^{max} takes only a finite number of values, and the sup becomes a max. Nevertheless, we can write

$$\begin{aligned} \Pi_n^{max}(f, g)(x) &= \sum_{I \in \mathcal{J}} \frac{1}{|I|} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle h_{I_n}(x) \cdot \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}}(x) = \\ &= \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle \tilde{h}_{I_n}(x), \end{aligned}$$

where $\tilde{h}_{I_n}(x) := h_{I_n}(x) \cdot \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}}$.

We start with the case $n = 0$; that is, the h_I are regular Haar functions, as opposed to being shifted n units. In this case, we have the following result:

Theorem 17.

$$\|\Pi^{max}(f, g)\|_s \lesssim \|f\|_p \|g\|_q,$$

given that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, and $2 \geq s > \frac{1}{2}$.

Proof. The functions f and g are so that $|f| \leq \mathbf{1}_F$, $|g| \leq \mathbf{1}_G$, where F and G are subsets of \mathbb{R} of finite measure. Also, $H \subset \mathbb{R}$ is so that $|H| = 1$. Now define

$$\Omega = \{x : \mathcal{M}(\mathbf{1}_F)(x) > c|F|\} \cup \{x : \mathcal{M}(\mathbf{1}_G)(x) > c|G|\}$$

to be the exceptional set, and $H' := H \setminus \Omega$. h will be any function with the property that $|h| \leq \mathbf{1}_{H'}$. We consider the trilinear form $\Lambda_{\Pi^{max}}$ associated with the

operator Π_{Π}^{max} :

$$\Lambda_{\Pi^{max}}(f, g, h) = \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle \langle h, \tilde{h}_I \rangle.$$

Note that all the intervals $I \in \mathcal{J}$ which are relevant for the trilinear form (i.e. for which $\langle h, \tilde{h}_I \rangle \neq 0$) will be those that intersect the support of h , and in consequence Ω^c . This implies that

$$\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_F \tilde{\chi}_I^M dx \lesssim \inf_{x \in I} \mathbf{1}_I(x) \cdot \mathcal{M}(\mathbf{1}_F)(x) \lesssim |F|.$$

Now we use Proposition 10:

$$|\Lambda_{\Pi^{max}}(f, g, h)| \lesssim (\text{size } {}_{\mathcal{J}}f)^{1-\theta_1} \cdot (\text{size } {}_{\mathcal{J}}g)^{1-\theta_2} \cdot (\widetilde{\text{size}} {}_{\mathcal{J}}h)^{1-\theta_3} \cdot (\text{energy } {}_{\mathcal{J}}f)^{\theta_1} \cdot (\text{energy } {}_{\mathcal{J}}g)^{\theta_2} \cdot (\text{energy } {}_{\mathcal{J}}h)^{\theta_3}.$$

Then we use the estimates that we have for *sizes* and *energies*:

$$\text{size } {}_{\mathcal{J}}f \lesssim \sup_{I \in \mathcal{J}} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \tilde{\chi}_I^M dx \lesssim \min\{1, |F|\}.$$

We take advantage of the fact that the intervals I intersect Ω^c . Similarly,

$$\text{size } {}_{\mathcal{J}}g \lesssim \sup_{I \in \mathcal{J}} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_G(x) \cdot \tilde{\chi}_I^M dx \lesssim \min\{1, |G|\}.$$

For the energies, we have

$$\text{energy } {}_{\mathcal{J}}f \lesssim \|f\|_1 \lesssim |F|, \quad \text{and} \quad \text{energy } {}_{\mathcal{J}}g \lesssim \|g\|_1 \lesssim |G|.$$

The size of h is similar to sizes appearing in [20] and in [25], which is only natural because in this case also we are dealing with a maximal operator. Following the definition of the size, and using John-Nirenberg, we have

$$\begin{aligned} \text{size } {}_{\mathcal{J}}h &:= \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|} \left\| \left(\sum_{I \subseteq I_0} \frac{|\langle h, \tilde{h}_I \rangle|^2}{|I|} \mathbf{1}_I \right) \right\|_{1,\infty} \simeq \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|^{1/2}} \left\| \left(\sum_{I \subseteq I_0} \frac{|\langle h, \tilde{h}_I \rangle|^2}{|I|} \mathbf{1}_I \right) \right\|_2 \simeq \\ &\simeq \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|^{1/2}} \left(\sum_{I \subseteq I_0} |\langle h, \tilde{h}_I \rangle|^2 \right)^{1/2}. \end{aligned}$$

In the last row, we dualize the ℓ^2 norm to obtain a more linear expression

$$\frac{1}{|I_0|^{1/2}} \left(\sum_{I \subseteq I_0} |\langle h, \tilde{h}_I \rangle|^2 \right)^{1/2} = \frac{1}{|I_0|^{1/2}} \sum_{I \subseteq I_0} a_I \langle h, \tilde{h}_I \rangle = \frac{1}{|I_0|^{1/2}} \langle h, \sum_{I \subseteq I_0} a_I \tilde{h}_I \rangle.$$

Recall that $\tilde{h}_I(x) = h_I(x) \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}}(x)$. For every $x \in I$, there exists a unique dyadic interval containing x , of length $2^{m(x)}$. We denote it $K(x)$. We note the following:

$$\sum_{I \subseteq I_0} a_I h_I(x) \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}} = \frac{1}{|K(x)|} \int_{K(x)} \sum_{I \subseteq I_0} a_I h_I(y) dy, \quad (3.3)$$

and this is actually bounded above by $\leq \mathcal{M} \left(\sum_{I \subseteq I_0} a_I h_I \right) (x)$.

The equality in 3.3 is explained by the following fact: whenever I' is an interval of length $< 2^{m(x)}$, then either it doesn't intersect $K(x)$ and so it doesn't have any contribution, or it is strictly contained in $K(x)$, and so the integral of $h_{I'}$ over $K(x)$ becomes 0. Hence only the intervals of length $\geq 2^{m(x)}$ play a role in the summation.

We once again accentuate that the localization of the operator Π^{max} will bring forth the similarities shared with other well-known operators, or compositions of several of these. In this case, it is the composition of the maximal operator, and a Haar multiplier. We conclude

$$\frac{1}{|I_0|^{1/2}} \left(\sum_{I \subseteq I_0} |\langle h, \tilde{h}_I \rangle|^2 \right)^{1/2} \lesssim \frac{1}{|I_0|^{1/2}} \|h \cdot \mathbf{1}_{I_0}\|_2 \|\mathcal{M} \left(\sum_{I \subseteq I_0} a_I h_I \right)\|_2 \lesssim \quad (3.4)$$

$$\lesssim \left(\frac{1}{|I_0|} \int_{\mathbb{R}} |h(x)|^2 \mathbf{1}_{I_0}(x) dx \right)^{1/2}. \quad (3.5)$$

The *energies* are, in some sense, the dual quantities for the *sizes*; the size ${}_j h$ can easily be estimated in L^2 , but it is much more difficult to compute in L^1 (however, due to the John-Nirenberg theorem, they are equivalent). The energy ${}_j h$ displays

the same behavior: in L^2 it behaves nicely, while for an L^1 estimate, one would need to study the following *maximal square function*:

$$f \mapsto \tilde{S}f(x) = \left(\sum_{I \in \mathcal{I}} \frac{\langle f, h_I \cdot \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}} \rangle}{|I|} \mathbf{1}_I(x) \right)^{1/2},$$

where $m : \mathbb{R} \rightarrow \mathbb{Z}$ is an arbitrary function.

Even though we don't obtain the optimal range for Π^{max} , instead of investigating \tilde{S} , we are going to use L^2 -normalized energy ${}_J h$. This is defined as

$$E_3(2) := \sup_{n \in \mathbb{Z}} 2^n \sup_{\mathbb{D}} \left(\sum_{I_0 \in \mathbb{D}} |I_0| \right)^{1/2},$$

where \mathbb{D} ranges over all collections of disjoint dyadic intervals I_0 with the property that

$$\frac{1}{|I_0|^{1/2}} \left\| \left(\sum_{I \subseteq I_0} \frac{\langle h, h_I \cdot \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}} \rangle}{|I|} \mathbf{1}_I(x) \right)^{1/2} \right\|_2 \geq 2^n.$$

The last condition is equivalent to $\left(\sum_{I \subseteq I_0} |\langle h, \tilde{h}_I \rangle|^2 \right)^{1/2} \geq |I_0|^{1/2} 2^n$.

Using this and 3.4,

$$\begin{aligned} E_3(2) &= \left(\sum_{I_0 \in \mathbb{D}} 2^{2n} |I_0| \right)^{1/2} \leq \left(\sum_{I_0 \in \mathbb{D}} \sum_{I \subseteq I_0} |\langle h, \tilde{h}_I \rangle|^2 \right)^{1/2} \lesssim \\ &\lesssim \left(\sum_{I_0 \in \mathbb{D}} \|h \cdot \mathbf{1}_{I_0}\|_2^2 \right)^{1/2} \leq \|h\|_2. \end{aligned}$$

For the very last step, we used the fact that the intervals $I_0 \in \mathbb{D}$ are all disjoint.

Instead of Proposition 10, however, we will need the following very similar result:

$$|\Lambda_{\Pi^{max}}(f, g, h)| \lesssim S_1^{1-\theta_1} S_2^{1-\theta_2} S_3^{1-2\theta_3} E_1^{\theta_1} E_2^{\theta_2} E_3(2)^{2\theta_3}, \quad (3.6)$$

where $\theta_1 + \theta_2 + \theta_3 = 1, 0 \leq \theta_1, \theta_2, \theta_3 < 1$. We will want $0 \leq \theta_3 \leq \frac{1}{2}$, which is an unfortunate restriction. In the end, this will impose $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} \geq \theta_1 + \theta_2 \geq \frac{1}{2}$, but

our main application requires $\Pi^{max} : L^2 \times L^2 \rightarrow L^1$, so this is acceptable. To get the full range, one would need to deal with the L^1 -adapted energy for h .

Recall that we have so far:

$$S_1 \lesssim \min\{1, |F|\}, \quad S_2 \lesssim \min\{1, |G|\}, \quad S_3 \lesssim \sup_{I_0 \in \mathcal{J}} \frac{1}{|I_0|^{1/2}} \|h \cdot \mathbf{1}_{I_0}\|_2 \lesssim 1$$

$$E_1 \lesssim |F|, \quad E_2 \lesssim |G|, \quad E_3(2) \lesssim \|h\|_2 \lesssim 1. \quad (3.7)$$

Thus $|\Lambda_{\Pi^{max}}(f, g, h)| \lesssim |F|^{\nu_1 + \theta_1} |G|^{\nu_2 + \theta_2}$, where $0 \leq \nu_1, \nu_2$, and $\frac{1}{p} \sim \nu_1 + \theta_1 \leq 1$, $\frac{1}{q} \sim \nu_2 + \theta_2 \leq 1$. Restricted weak type interpolation theory implies the desired conclusion. \square

Now we will prove a similar result for the $\Pi_{\mathfrak{n}}^{max}$, whose last component is represented by a shifted Haar function.

Theorem 18.

$$\|\Pi_{\mathfrak{n}}^{max}(f, g)\|_s \lesssim (\log \langle \mathfrak{n} \rangle)^7 \|f\|_p \|g\|_q$$

for any $1 < p, q \leq \infty$, $\frac{1}{2} < s \leq 2$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

Proof. The trilinear form associated with this operator is

$$\Lambda_{\Pi_{\mathfrak{n}}^{max}}(f, g, h) := \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle \langle h, \tilde{h}_{I_{\mathfrak{n}}} \rangle.$$

The third component is complicated because of two reasons: first, we have to deal with functions $\tilde{h}_I = h_I \cdot \mathbf{1}_{\{x: 2^{m(x)} \leq |I|\}}$, which won't necessarily form an orthogonal system anymore. And second, because of the shifted Haar functions. To simplify somehow this operator, we denote $\tilde{\mathcal{J}}$ the shifted intervals of I :

$$\tilde{\mathcal{J}} := \{I : I_{-\mathfrak{n}} \in \mathcal{J}\} = \{I_{\mathfrak{n}} : I \in \mathcal{J}\}.$$

The boundedness of the operator $\Pi_{\mathfrak{n}}^{max}$ is going to be independent of the collection of intervals, as one would expect, so we can regard $\Pi_{\mathfrak{n}}^{max}$ as

$$\Pi_{\mathfrak{n}}^{max}(f, g) := \sum_{I \in \tilde{\mathcal{J}}} \frac{1}{|I|^{1/2}} \langle f, \varphi_{I_{\mathfrak{n}}} \rangle \langle g, \psi_{I_{\mathfrak{n}}} \rangle \tilde{h}_I(x).$$

Here we assume F, G and H are sets of finite measure, and $|H| = 1$. We construct the exceptional set:

$$\Omega := \{x : \mathcal{M}^{\mathfrak{n}}(\mathbf{1}_F) > C \log \langle \mathfrak{n} \rangle |F|\} \cup \bigcup_{l=0}^{\log \langle \mathfrak{n} \rangle} \{x : \mathcal{M}^{2^l}(\mathbf{1}_G) > C (\log \langle \mathfrak{n} \rangle)^2 |G|\}.$$

The rather strange condition for the exceptional set is motivated by the following fact: if I' is a fixed dyadic interval, then for any subinterval $I'' \subseteq I'$ its translate $I''_{\mathfrak{n}}$ is going to be contained in one of the $\log \langle \mathfrak{n} \rangle$ translates of I' . We denote these translates by I_0^l , where $0 \leq l \leq \log \langle \mathfrak{n} \rangle$, and I_0^l is in fact $I_{0,2^l} := I_0 + 2^l |I_0|$. Not only will we need control of the shifted maximal operator $\mathcal{M}^{\mathfrak{n}}$ of $\mathbf{1}_G$, but also of the shifted maximal operator \mathcal{M}^{2^l} , for $0 \leq l \leq \log \langle \mathfrak{n} \rangle$.

Then $H' := H \setminus \Omega$ is a major subset of H , contained in Ω^c , because $|\Omega| \ll 1$ for a constant C sufficiently large. f, g and h are functions so that

$$|f| \leq \mathbf{1}_F, \quad |g| \leq \mathbf{1}_G, \quad |h| \leq \mathbf{1}_{H'}.$$

Proposition 3.6 is going to be useful again, because the last component is identical to the previous one. We have

$$|\Lambda_{\Pi_{\mathfrak{n}}^{max}}(f, g, h)| \lesssim S_1^{1-\theta_1} S_2^{1-\theta_2} S_3^{1-2\theta_3} E_1^{\theta_1} E_2^{\theta_2} E_3(2)^{2\theta_3}$$

and now we have to estimate the *sizes* and *energies* associated with the functions f and g . These will resemble localized shifted maximal operators and shifted square functions.

For the first function, we have

$$\begin{aligned} \text{size } \tilde{J}f &:= \sup_{I \in \tilde{J}} \frac{|\langle f, \varphi_{I_n} \rangle|}{|I|^{1/2}} \lesssim \sup_{I \in \tilde{J}} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_F(x) \tilde{\chi}_{I_n}(x) dx \lesssim \\ &\lesssim \sup_{I \in \tilde{J}} \mathbf{1}_I(x) \cdot \mathcal{M}_n(\mathbf{1}_F)(x) \lesssim \log \langle n \rangle |F|. \end{aligned}$$

The energy quantity attains its maximum for some $m \in \mathbb{Z}$, and a collection \mathbb{D} of disjoint dyadic intervals, for each of which we have

$$2^m \mathbf{1}_I(x) \leq \frac{|\langle f, \varphi_{I_n} \rangle|}{|I|^{1/2}} \mathbf{1}_I(x) \lesssim \mathbf{1}_I(x) \mathcal{M}_n(\mathbf{1}_F)(x).$$

Then we claim that $\text{energy } \tilde{J}h \lesssim \log \langle n \rangle |F|$:

$$\text{energy } \tilde{J}h = 2^m \sum_{I_0 \in \mathbb{D}} |I_0| = \left\| \sum_{I_0 \in \mathbb{D}} 2^m \mathbf{1}_{I_0} \right\|_{1,\infty} \lesssim \|\mathcal{M}_n \mathbf{1}_F\|_{1,\infty} \lesssim \log \langle n \rangle |F|.$$

The *sizes* and *energies* for g are slightly more complicated, but one can deal with them due to the boundedness of the shifted square function. In section 7.1 we proved the boundedness of \mathcal{S}_n on L^p , with $1 < p < \infty$, but in this case we need a more specific estimate: $\mathcal{S}_n : L^1 \rightarrow L^{1,\infty}$. Our methods are fitted for the $p \geq 2$ situation, but we will use the result in [6], which also provides a $\log \langle n \rangle$ operatorial norm.

Another important remark is that, even though $\|\cdot\|_{1,\infty}$ is not a norm, it is a quasi-norm and one can still easily prove

$$\|f_1 + \dots + f_N\|_{1,\infty} \leq N (\|f_1\|_{1,\infty} + \dots + \|f_N\|_{1,\infty}).$$

We will be using this with $N = \log \langle n \rangle$. Following the definition, we have

$$\text{size } \tilde{J}g \sup_{I_0 \in \tilde{J}} = \frac{1}{|I_0|} \left\| \left(\sum_{I \subseteq I_0} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty}.$$

We fix an interval $I_0 \in \tilde{\mathcal{J}}$; already we've seen that if $I \subseteq I_0$, then I_n will be in one of $\log\langle n \rangle$ translates of I_0^l , where $0 \leq l \leq \log\langle n \rangle$. Then

$$\frac{1}{|I_0|} \left\| \left(\sum_{I \subseteq I_0} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty} \lesssim \log\langle n \rangle \sum_{l=0}^{\log\langle n \rangle} \frac{1}{|I_0|} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I_n \subseteq I_0^l}} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty}.$$

The last expression is going to be treated as a classical *size*; we just repeat the argument from [6]. For any $0 \leq l \leq \log\langle n \rangle$, we partition \mathbb{R} into intervals K_m of length $|I_0^l|$, where $m \in \mathbb{Z}$, and $K_m := I_0^l$.

$$\begin{aligned} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I_n \subseteq I_0^l}} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty} &\lesssim \sum_{|m| \leq 2} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I_n \subseteq I_0^l}} \frac{|\langle g \cdot \mathbf{1}_{K_m}, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty} + \\ &+ \sum_{|m| \geq 3} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I_n \subseteq I_0^l}} \frac{|\langle g \cdot \mathbf{1}_{K_m}, \psi_{I_n} \rangle|^2}{|I_n|} \mathbf{1}_I \right)^{1/2} \right\|_1 \end{aligned}$$

For the first part, we use the $L^1 \rightarrow L^{1,\infty}$ boundedness of the shifted square function \mathcal{S}_n , while for the second one we take advantage of the decay of the ψ_I functions.

In the end, all these will imply that

$$\text{size } \mathcal{J}g \lesssim \sup_{I_0 \in \tilde{\mathcal{J}}} \sum_{l=0}^{\log\langle n \rangle} \frac{1}{|I_0^l|} \int_{\mathbb{R}} \mathbf{1}_G \tilde{\chi}_{I_0^l}^M dx \lesssim (\log\langle n \rangle)^3 \min\{1, |G|\}.$$

For the energy, we have

$$\begin{aligned}
\text{energy } \tilde{g} &= 2^\# \sum_{I_0 \in \mathbb{D}} |I| = \left\| \sum_{I_0 \in \mathbb{D}} 2^\# \mathbf{1}_{I_0} \right\|_{1,\infty} \leq \\
&\leq \left\| \sum_{I_0 \in \mathbb{D}} \frac{1}{|I_0|} \left\| \left(\sum_{I \subseteq I_0} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty} \mathbf{1}_{I_0} \right\|_{1,\infty} \lesssim \\
&\lesssim (\log \langle n \rangle)^2 \sum_{l=0}^{\log \langle n \rangle} \left\| \sum_{I_0 \in \mathbb{D}} \frac{1}{|I_0|} \left\| \left(\sum_{\substack{I \subseteq I_0 \\ I_n \subset I'_0}} \frac{|\langle g, \psi_{I_n} \rangle|^2}{|I|} \mathbf{1}_I \right)^{1/2} \right\|_{1,\infty} \mathbf{1}_{I_0} \right\|_{1,\infty} \lesssim \\
&\lesssim (\log \langle n \rangle)^3 \sum_{l=0}^{\log \langle n \rangle} \left\| \sum_{I_0 \in \mathbb{D}} \frac{1}{|I_0|} \|g \cdot \tilde{\chi}_{I'_0}^M\|_1 \mathbf{1}_{I_0} \right\|_{1,\infty} \lesssim \\
&\lesssim (\log \langle n \rangle)^3 \sum_{l=0}^{\log \langle n \rangle} \left\| \sum_{I_0 \in \mathbb{D}} \mathcal{M}_l(|g|) \cdot \mathbf{1}_{I_0} \right\|_{1,\infty} \lesssim \\
&\lesssim (\log \langle n \rangle)^5 \sum_{l=0}^{\log \langle n \rangle} \log \langle m \rangle |G| \leq (\log \langle n \rangle)^7 |G|.
\end{aligned}$$

Here we are not rigorous about the powers of $\log \langle n \rangle$; for the Rubio de Francia paraproduct theorem, we just need them to decay slower than a power of n . We have

$$\begin{aligned}
S_1 &\lesssim \min \log \langle n \rangle \{1, |F|\}, \quad S_2 \lesssim (\log \langle n \rangle)^3 \min \{1, |G|\}, \quad S_3 \lesssim 1 \\
E_1 &\lesssim \log \langle n \rangle |F|, \quad E_2 (\log \langle n \rangle)^5 |G|, \quad E_3(2) \lesssim \|h\|_2 \lesssim 1.
\end{aligned} \tag{3.8}$$

Using all these we conclude

$$|\Lambda_{\Gamma_n^{\max}}(f, g, h)| \lesssim (\log \langle n \rangle)^7 |F|^{\nu_1 + \theta_1} |G|^{\nu_2 + \theta_2},$$

where $0 \leq \nu_1, \nu_2 < 1$, and $\frac{1}{p} \sim \nu_1 + \theta_1 \leq 1$, $\frac{1}{q} \sim \nu_2 + \theta_2 \leq 1$. Least but not last, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} \geq \theta_1 + \theta_2 \geq \frac{1}{2}$. \square

3.2 Rubio de Francia for Paraproducts

Recall that a paraproduct is defined by the formula

$$\Pi(f, g)(x) = \sum_{I \in \mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \varphi_I \rangle \langle g, \psi_I \rangle \psi_I(x),$$

while the operator we were initially interested in is

$$T_r(f, g)(x) = \left(\sum_{k=1}^N \left| \int_{a_k < \xi_1 < \xi_2 < b_k} \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^r \right)^{\frac{1}{r}}.$$

The operator T_r , a bilinear version of Rubio de Francia's square function, is associated to the frequency levels $a_k < \xi_1 < \xi_2 < b_k$, where $1 \leq k \leq N$. For each k , we restrict our attention to the squares in the Whitney decomposition with $\xi_1 \sim a_k$. The frequency intervals for ξ_2 become lacunary with respect to a_k . In this case, knowing the diameter of a square implies knowing the position of the square, and the associated bilinear operator becomes a paraproduct:

$$\Pi_k(f, g)(x) = \sum_{I \in \mathcal{D}_k} \frac{1}{|I|^{1/2}} \langle f, \varphi_I^k \rangle \langle g, \psi_I^k \rangle \psi_I^k(x).$$

Each φ_I^k is L^2 -normalized, adapted to $I \in \mathcal{D}_k$, and $\hat{\varphi}_I^k$ is supported on $[a_k - \frac{1}{2|I|}, a_k + \frac{1}{2|I|}]$, and $\equiv 1$ on $\frac{9}{10}[a_k - \frac{1}{2|I|}, a_k + \frac{1}{2|I|}]$. Similarly, ψ_I^k is L^2 -normalized, adapted to I ; $\hat{\psi}_I^k$ is supported on $[a_k + \frac{1}{|I|}, a_k + \frac{2}{|I|}]$, and $\equiv 1$ on $\frac{9}{10}[a_k + \frac{1}{|I|}, a_k + \frac{2}{|I|}]$.

We consider the operator corresponding to T_1 ; this is an instance of an l^1 -valued paraproduct:

$$T(f, g)(x) = \sum_{k=1}^N |\Pi_k(f, g)(x)|.$$

Theorem 19. *The operator T defined above is bounded from $L^p \times L^q$ into L^s , for any $p, q \geq 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$:*

$$\|T(f, g)\|_s \lesssim \|f\|_p \|g\|_q.$$

Dualizing the l^1 norm, we can regard T as

$$T(f, g) = \sum_{k=1}^N \epsilon_k(x) \sum_{I \in \mathcal{D}_k} \frac{\langle f, \varphi_I^k \rangle}{|I|^{1/2}} \langle g, \psi_I^k \rangle \psi_I^k(x).$$

Here for a.e. x , $\sup_k |\epsilon_k(x)| = 1$.

We will need the following *sizes* for f, g , and h . The tree estimate below will shed some light on these definitions.

Definition 24. For f and g the sizes are defined in a similar way:

$$\overline{\text{size}}_{I_T}(f) = \sup_{\substack{I \in \mathcal{J} \\ I \subseteq I_T}} \left(\frac{1}{|I|} \int_{\mathbb{R}} |RF(f)(x)|^2 \cdot \tilde{\chi}_I^2(x) dx \right)^{1/2}.$$

On the other hand, the size for h is slightly more complicated, but it resembles the quantity appearing in the estimates for the Rubio de Francia square function.

Definition 25. Given an interval I_T and a collection \mathcal{J} of dyadic intervals, we denote $\mathcal{J}^+(I_T) := \{I' \subseteq I_T : \text{there exists some } I \in \mathcal{J}, I \subseteq I', \text{ so that } I \subseteq I' \subseteq I_T\}$. Then we define

$$\overline{\text{size}}_{I_T}(h) = \sup_{I' \in \mathcal{J}^+(I_T)} \sup_{I \subseteq I' \subseteq I_T} \frac{1}{|3I'|} \int_{\mathbb{R}} |h(x)| \cdot \tilde{\chi}_{3I'}(x) dx.$$

Remark 7. The paraproduct Π_k is just a translate of the classical paraproduct Π . Also, note that

$$\langle f, \varphi_I^k \rangle = \langle \hat{f}, \hat{\varphi}_I^k \rangle = \langle \hat{f} \cdot \mathbf{1}_{[\frac{3a_k-b_k}{2}, \frac{a_k+b_k}{2}]} , \hat{\varphi}_I^k \rangle = \langle e^{-2\pi i x a_k} \hat{f} \cdot \mathbf{1}_{[\frac{3a_k-b_k}{2}, \frac{a_k+b_k}{2}]} , \hat{\varphi}_I \rangle := \langle f_k, \varphi_I \rangle.$$

$$\text{Here } f_k(x) = e^{-2\pi i x a_k} \cdot f * \check{\mathbf{1}}_{[\frac{3a_k-b_k}{2}, \frac{a_k+b_k}{2}]}(x).$$

$$\text{Identically, } \langle g, \psi_I^k \rangle = \langle g_k, \psi_I \rangle, \text{ where } g_k(x) = e^{-2\pi i x a_k} \cdot g * \check{\mathbf{1}}_{[a_k, b_k]}(x).$$

We can rewrite T as

$$T(f, g) = \sum_k \epsilon_k(x) \sum_{I \in \mathcal{J}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle e^{2\pi i a_k x} \psi_I(x).$$

The trilinear form associated to the operator T , is given by

$$\langle T(f, g), h \rangle = \int_{\mathbb{R}} T(f, g)(x) \bar{h}(x) dx = \int_{\mathbb{R}} \sum_k \sum_{I \in \mathcal{J}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx,$$

where $h_k(x) = \epsilon_k(x) e^{2\pi i a_k x} h(x)$.

Lemma 5 (Tree/ local estimate). *Fix an interval I_T ; then*

$$\left| \int_{\mathbb{R}} \sum_k \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_T}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx \right| \lesssim \overline{\text{size}}_{I_T}(f) \cdot \overline{\text{size}}_{I_T}(g) \cdot \overline{\text{size}}_{I_T}(g) \cdot |I_T|.$$

Proof. Let \mathcal{J} be the collection of maximal dyadic intervals J with the property that no $I \subset 3J$. This forms a partition of \mathbb{R} . Also, if \tilde{J} denotes the dyadic parent of J , the maximality condition implies that for any $J \in \mathcal{J}$, there exists an interval $I(J) \in \mathcal{J}$, $I(J) \subset 3\tilde{J}$. We can rewrite

$$\int_{\mathbb{R}} \sum_k \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_T}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx = \sum_{J \in \mathcal{J}} \int_J \sum_k \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_T}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx$$

and the task is now to estimate the average over J . There are three possibilities:

- (i) $J \subset I_T$ and $|J| \leq |I|$ for the intervals I around J
- (ii) $J \subset 3I_T$, but $|J| > |I|$ for the intervals I sufficiently far away from J
- (iii) J is far away from I_T , but $3\tilde{J} \cap I_T \neq \emptyset$.

The local estimate above splits as

$$\sum_{\substack{J \in \mathcal{J} \\ J \subseteq 3I_T}} \int_J \sum_k \sum_{\substack{I \subseteq I_T \\ |I| > |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx + \quad (\text{I})$$

$$+ \sum_{\substack{J \in \mathcal{J} \\ J \subseteq 3I_T}} \int_J \sum_k \sum_{\substack{I \subseteq I_T \\ |I| \leq |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx + \quad (\text{II})$$

$$+ \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \int_J \sum_k \sum_{\substack{I \subseteq I_T \\ |I| \leq |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx. \quad (\text{III})$$

The behavior of each term is distinct; (I) gives the main contribution, and will be bounded by a maximal version of a paraproduct. (II) is a sort of an $l^\infty(I \subseteq I_T)$ estimate, while (III) is easily estimated using the fast decay of the φ_I and ψ_I .

We need to introduce a partition of the collection of intervals (I, J) , the importance of which will become visible soon. For any $l > 0$ and $n \geq 0$, define

$$(\mathcal{I} \times \mathcal{J})_n^l := \{(I, J) : |I| = 2^l |J| \text{ and } \frac{\text{dist}(I, J)}{|I|} \sim n\}.$$

Also, \mathcal{J}_n will denote the set of intervals J so that $(I, J) \in (\mathcal{I} \times \mathcal{J})_n^l$ for some $l > 0$ and some interval I . The first term (I) becomes

$$\sum_{n \geq 1} \sum_{\substack{J \in \mathcal{J}_n \\ J \subseteq 3I_T}} \int_J \sum_k \sum_{\substack{I \subseteq I_T \\ |I| > |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx.$$

Fix n, k and an interval $J \in \mathcal{J}_n$; we are summing over intervals $I \subseteq I_T, |I| > |J|$. Morally speaking, because ψ_I decays fast away from I and because J is smaller than the intervals I appearing in the sum, we can regard $|\psi_I|$ as being constantly equal to $(\log \langle n \rangle)^{-100}$ on J . Also, because of the way J was defined, there exist an interval $I(J)$ so that $I(J) \subseteq 3\tilde{J}$. Then both $I(J)$ and J are going to be contained in $3J'$, where J' is \tilde{J} , or its translate to the left or to the right.

$$\begin{aligned} & \left| \int_J \sum_{\substack{I \subseteq I_T \\ (I, J) \in (\mathcal{I} \times \mathcal{J})_n}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx \right| \lesssim \\ & \lesssim \frac{1}{|J'|} \int_{J'} |h(x)| dx \cdot |J| \cdot \sup_{x \in J} \left| \sum_{\substack{I \subseteq I_T \\ (I, J) \in (\mathcal{I} \times \mathcal{J})_n}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) \right| \lesssim \\ & \lesssim \frac{1}{|J'|} \int_{J'} |h(x)| dx \cdot |J| \cdot \sup_{x \in J} \left| \sum_{\substack{I \subseteq I_T \\ (I, J) \in (\mathcal{I} \times \mathcal{J})_n}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) \right| \lesssim \\ & \lesssim \frac{1}{\langle n \rangle^{100}} \frac{1}{|J'|} \int_{J'} |h(x)| dx \cdot |J| \cdot \sup_{x \in J} \sum_{\substack{I \subseteq I_T \\ (I, J) \in (\mathcal{I} \times \mathcal{J})_n}} \frac{|\langle f_k, \varphi_I \rangle|}{|I|^{1/2}} |\langle g_k, \psi_I \rangle| |h_{I_n}(x)| \end{aligned}$$

The functions $h_{I_n}(x)$ appearing above are the shifted Haar functions, where $h_{I_0}(x) = h_I(x) = \frac{1}{|I|^{1/2}}(\mathbf{1}_{I_{\text{left}}} - \mathbf{1}_{I_{\text{right}}})$. Because h_I is constant on its left and right halves, and J is fully contained inside either of these,

$$\sum_{\substack{I \subseteq I_T \\ (I, J) \in (\mathcal{I} \times \mathcal{J})_n}} \frac{|\langle f_k, \varphi_I \rangle|}{|I|^{1/2}} |\langle g_k, \psi_I \rangle| |h_{I_n}(x)|$$

is constant on J . It is bounded pointwise by the maximal paraproduct operator, with absolute values inside the summation:

$$\Pi_n^{\max}(f_k, g_k)(x) := \sup_{m \in \mathbb{Z}} \sum_{\substack{I \in \mathcal{J} \\ |I| \geq 2^m}} \frac{|\langle f_k, \varphi_I \rangle|}{|I|^{1/2}} |\langle g_k, \psi_I \rangle| |h_{I_n}(x)|.$$

For the trilinear form associated with the discretized maximal paraproduct, there is no difference in the method of proof if we insert absolute values inside the summation; now we don't have a trilinear form, but the exterior function is h_{I_n} , a shifted Haar function, which takes the value $\frac{1}{\sqrt{|I|}}$ or $-\frac{1}{\sqrt{|I|}}$ on two dyadic intervals. We can write the above operator as a linear combination of four operators for which all the terms appearing in the summation are ≥ 0 .

Recall that we have the following estimates for the maximal paraproduct:

$$\|\Pi_n^{\max}(f, g)\|_s \lesssim (\log \langle n \rangle)^7 \|f\|_p \|g\|_q,$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}, \frac{1}{2} < s \leq 2$. We will be using the $L^2 \times L^2 \rightarrow L^1$ boundedness of this operator.

We can estimate (I) by

$$\begin{aligned}
& \sum_k \sum_n \sum_{J \in \mathcal{J}_n} \frac{1}{|J'|} \int_{J'} |h(x)| dx \cdot \frac{1}{\log \langle n \rangle^{100}} \int_J |\Pi_n^{\max}(f_k, g_k)(x)| dx \lesssim \\
& \lesssim \sum_k \sum_n \frac{1}{\langle n \rangle^{100}} \left(\sup_{I \subseteq I_T} \sup_{I' \subseteq I} \frac{1}{|I'|} \int_{\mathbb{R}} |h(x)| \tilde{\chi}_{I'}(x) dx \right) \cdot \|\Pi_n^{\max}(f_k, g_k)\|_{L^1(I_T)} \lesssim \\
& \lesssim \sum_n \frac{\log \langle n \rangle^7}{\langle n \rangle^{100}} \overline{\text{size}}_{I_T}(h) \sum_k \|f_k \cdot \tilde{\chi}_{I_T}\|_2 \cdot \|g_k \cdot \tilde{\chi}_{I_T}\|_2 \lesssim \\
& \lesssim \overline{\text{size}}_{I_T}(h) \cdot \overline{\text{size}}_{I_T}(f) \cdot \overline{\text{size}}_{I_T}(g) \cdot |I_T|.
\end{aligned}$$

Here we need argue why $\|\Pi_n^{\max}(f_k, g_k)\|_{L^1(I_T)} \leq (\log \langle n \rangle)^7 \|f_k \cdot \tilde{\chi}_{I_T}\|_2 \cdot \|g_k \cdot \tilde{\chi}_{I_T}\|_2$, which is a more localized result than theorem 18. If we look closely at this paraproduct, we notice that all intervals $I \subseteq I_T$, and also $I_n \subseteq I_T$. So if f_k was to be supported on $2^{L+1}I_T \setminus 2^L I_T$, then the fast decay of the φ_I would account for an extra 2^{-ML} factor (appearing from the size estimates in the proof of Thm. 18).

We proceed with estimating (II):

$$\begin{aligned}
& \left| \sum_{k=1}^N \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \int_J \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) \cdot h_k(x) dx \right| = \\
& = \left| \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \int_J \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \sum_{k=1}^N \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) \cdot h_k(x) dx \right| \lesssim \\
& \lesssim \left| \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \int_J \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\sum_{k=1}^N |\langle f_k, \varphi_I \rangle|^2 \right)^{1/2} \left(\sum_{k=1}^N |\langle g_k, \psi_I \rangle|^2 \right)^{1/2} \left| \sup_k h_k(x) \cdot \frac{1}{|I|^{1/2}} \psi_I(x) dx \right| \right| \lesssim \\
& \lesssim \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \int_J \sup_I \left(\frac{1}{|I|} \sum_{k=1}^N |\langle f_k, \varphi_I \rangle|^2 \right)^{1/2} \cdot \sup_I \left(\frac{1}{|I|} \sum_{k=1}^N |\langle g_k, \psi_I \rangle|^2 \right)^{1/2} \cdot \\
& \quad \cdot \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} |I| |h(x)| \cdot \frac{1}{|I|} \left(\frac{\text{dist}(I, x)}{|I|} \right)^{-100} dx
\end{aligned}$$

In this case, the role of a tree is played by a tile, and that of the vectorial tree by

a stack of tiles.

$$\begin{aligned} & \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \int_J \sup_I \left(\frac{1}{|I|} \sum_{k=1}^N |\langle f_k, \varphi_I \rangle|^2 \right)^{1/2} \cdot \sup_I \left(\frac{1}{|I|} \sum_{k=1}^N |\langle g_k, \psi_I \rangle|^2 \right)^{1/2} \\ & \cdot \sup_I \frac{1}{|I|} \int_{\mathbb{R}} |h(x)| \cdot \tilde{\chi}_I(x) dx \cdot \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-50} \cdot |I|. \end{aligned}$$

This is exactly the estimate we want, provided

$$\sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-50} \cdot |I| \lesssim |I_T|.$$

We estimate this in the following way:

$$\begin{aligned} & \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \sum_{l>0} \sum_{|I|=2^{-l}|J|} \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-30} \cdot \left(\frac{\text{dist}(x, I)}{|J|} \cdot \frac{|J|}{|I|} \right)^{-20} 2^{-l}|J| \lesssim \\ & \lesssim \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} \sum_{l>0} 2^{-20l}|J| \cdot \sum_{|I|=2^{-l}|J|} \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-30} \lesssim \\ & \lesssim \sum_{\substack{J \in \mathcal{J} \\ J \subset 3I_T}} |J| \lesssim 3|I_T|. \end{aligned}$$

Here we used the fact that the intervals J are all mutually disjoint and contained in $3I_T$.

The last term (III), we consider intervals J so that $|J| > |I_T|$; this will imply that the intervals are situated away from I_T , and we will try to take advantage of the fast decay of ψ_I s on the intervals J . Fix $1 \leq k \leq N$.

$$\begin{aligned} & \left| \sum_{\substack{J \in \mathcal{J} \\ |I_T| < |J|}} \int_J \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \frac{\langle f_k, \varphi_I \rangle}{|I|^{1/2}} \langle g_k, \psi_I \rangle \psi_I(x) h_k(x) dx \right| \lesssim \\ & \sum_{\substack{J \in \mathcal{J} \\ |I_T| < |J|}} \int_J \sup_I |\langle f_k, \varphi_I \rangle| \cdot \sup_I |\langle g_k, \psi_I \rangle| \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \frac{1}{|I|} \cdot |h(x)| \cdot \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-50} dx \end{aligned}$$

For any $I \subseteq I_T$, $|\langle f_k, \varphi_I \rangle| \lesssim \|f \cdot \tilde{\chi}_{I_T}\|_2$, because φ_I is L^2 normalized and decays much faster than $\tilde{\chi}_{I_T}$ outside I_T . Similarly, $|\langle g_k, \psi_I \rangle| \lesssim \|g \cdot \tilde{\chi}_{I_T}\|_2$.

$$\int_J \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \frac{1}{|I|} \cdot |h(x)| \cdot \left(\frac{\text{dist}(x, I)}{|I|} \right)^{-100} dx$$

is bounded above by

$$\left(\sup_I \frac{1}{|I|} \int_{\mathbb{R}} |h(x)| \tilde{\chi}_I(x) dx \right) \cdot \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\frac{\text{dist}(J, I)}{|I|} \right)^{-50}.$$

The first term is even nicer than the “size” of h that we were seeking to get; so the only thing left prove is that

$$\sum_{J \in \mathcal{J}} \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\frac{\text{dist}(J, I)}{|I|} \right)^{-50} \lesssim 1.$$

Then

$$\begin{aligned} \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \sum_{\substack{I \subseteq I_T \\ |I| < |J|}} \left(\frac{\text{dist}(J, I)}{|I|} \right)^{-50} &\lesssim \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \sum_{l > 0} \sum_{|I| = 2^{-l}|I_T|} \left(\frac{\text{dist}(J, I)}{|I|} \right)^{-50} \lesssim \\ &\lesssim \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \sum_{l > 0} \left(\frac{\text{dist}(J, I_T)}{|I|} \right)^{-40} \lesssim \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \sum_{l > 0} \left(\frac{\text{dist}(J, I_T)}{|I_T|} \cdot \frac{|I_T|}{|I|} \right)^{-40} \lesssim \\ &\lesssim \sum_{\substack{J \in \mathcal{J} \\ |J| > |I_T|}} \sum_{l > 0} 2^{-40l} \left(\frac{\text{dist}(I_T, J)}{|I_T|} \right)^{-40} \lesssim 1. \end{aligned}$$

□

3.2.1 Proof of Rubio de Francia theorem for Paraproducts

We proceed with the proof of Theorem 19; that is, using restricted weak type estimates, we will prove that

$$T(f, g)(x) = \sum_{k=1}^N |\Pi_k(f, g)(x)| =$$

is a bounded operator from $L^p \times L^q \rightarrow L^s$ whenever $p, q \geq 2, \frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

Start with F, G and H sets of finite measure, with $|H| = 1$. Let $|f| \leq \mathbf{1}_G, |g| \leq H$.

Define the exceptional set

$$\Omega = \{x : \mathcal{M}(|\text{RF}(f)|^2) > C\|f\|_2^2\} \cup \{x : \mathcal{M}(|\text{RF}(g)|^2) > C\|g\|_2^2\}$$

and set $H' = H \setminus \Omega$; for C large enough $|H'| \simeq |H|$:

$$\begin{aligned} |\{x : \mathcal{M}(|\text{RF}(f)|^2) > C\|f\|_2^2\}| &\leq (C\|f\|_2^2)^{-1} \|\mathcal{M}(|\text{RF}(f)|^2)\|_{1,\infty} \lesssim \\ &\lesssim (C\|f\|_2^2)^{-1} \| |\text{RF}(f)|^2 \|_1 = (C\|f\|_2^2)^{-1} \|\text{RF}(f)\|_2^2 \lesssim C \ll 1. \end{aligned}$$

Similarly for g , therefore we can conclude $|\Omega| \ll 1$, and H' is a major subset.

In this case, the exceptional set depends on the functions f and g , but this is okay in the Banach triangle (i.e. $1 < p, q, s < \infty$). Our estimates will imply that

$$\|T(f, g)\|_{s,\infty} \leq C\|f\|_p\|g\|_q,$$

and a bilinear Marcinkiewicz interpolation theorem as in [29] will imply the desired conclusion.

Let $\mathcal{J}_d = \{I \in \mathcal{J} : 1 + \frac{\text{dist}(\Omega^C, I)}{|I|} \sim 2^d\}$, for all $d \geq 0$ integer and clearly it is enough to show

$$\Lambda_{\mathcal{J}_d}(f, g, h) \lesssim 2^{-10d} |E|^{1/p_1} \cdot |F|^{1/p_2} \cdot |G|^{1/s'}.$$

For every $n_1 \in \mathbb{Z}$, define

$$\mathcal{J}_d^{n_1} := \left\{ I_0 \in \mathcal{J}_d : \frac{\|\text{RF}(f) \cdot \tilde{\chi}_{I_0}\|_2}{|I_0|^{1/2}} \sim 2^{-n_1} \quad \text{and } I_0 \text{ is maximal with this property} \right\}$$

In particular, the intervals in $\mathcal{J}_d^{n_1}$ are disjoint. Define similarly $\mathcal{J}_d^{n_2}$ and also

$$\mathcal{J}_d^{n_3} := \left\{ I_0 \in \mathcal{J}_d : \frac{|\langle h, \tilde{\chi}_{I_0}^M \rangle|}{|I_0|} \sim 2^{-n_3} \quad \text{and } I_0 \text{ is maximal with this property} \right\}$$

Then denote $\mathcal{J}_d^{n_1, n_2, n_3} = \mathcal{J}_d^{n_1} \cap \mathcal{J}_d^{n_2} \cap \mathcal{J}_d^{n_3}$.

$$\begin{aligned}
\Lambda_{\mathcal{J}_d}(f, g, h) &= \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}_d^{n_1, n_2, n_3}} \sum_{I \subseteq I_0} \sum_{k=1}^N \frac{1}{|I|^{1/2}} \langle f_k, \phi_I \rangle \langle g_k, \psi_I \rangle \langle h_k, \psi_I \rangle \lesssim \\
&\lesssim \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}_d^{n_1, n_2, n_3}} \text{size}_C^{\mathcal{J}_0}(h) \cdot \|\text{RF}(f) \cdot \tilde{\chi}_{I_0}\|_2 \cdot \|\text{RF}(g) \cdot \tilde{\chi}_{I_0}\|_2 \lesssim \\
&\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1} 2^{-n_2} 2^{-n_3} \sum_{I_0 \in \mathcal{J}_d^{n_1, n_2, n_3}} |I_0|
\end{aligned}$$

We will estimate $\sum_{I_0 \in \mathcal{J}_d^{n_1, n_2, n_3}} |I_0|$ in three different ways and interpolate between these estimates.

$$\sum_{I_0 \in \mathcal{J}_d^{n_1, n_2, n_3}} |I_0| \leq \sum_{I_0 \in \mathcal{J}_d^{n_1}} |I_0| = \left\| \sum_{I_0 \in \mathcal{J}_d^{n_1}} \chi_{I_0} \right\|_1$$

because all the intervals in $\mathcal{J}_d^{n_1}$ are mutually disjoint. Also, for each of them

$$\|\text{RF}(f) \tilde{\chi}_I\|_2 \sim |I|^{1/2} 2^{-n_1},$$

which is equivalent to

$$\frac{1}{|I|} \int |\text{RF}(f)|^2 \tilde{\chi}_I^2 dx \sim 2^{-2n_1}.$$

The LHS is $\lesssim \mathcal{M}(|\text{RF}(f)|^2)$ on I .

$$\begin{aligned}
\left\| \sum_{I \in \mathcal{J}_d^{n_1}} \chi_I \right\|_1 &= \left\| \sum_{I \in \mathcal{J}_d^{n_1}} \chi_I \right\|_{1, \infty} \lesssim 2^{2n_1} \left\| \sum_{I \in \mathcal{J}_d^{n_1}} \left(\frac{1}{|I|} \int |\text{RF}(f)|^2 \tilde{\chi}_I^2 dx \right) \chi_I \right\|_{1, \infty} \lesssim \\
&\lesssim 2^{2n_1} \|\mathcal{M}(|\text{RF}(f)|^2)\|_{1, \infty} \lesssim 2^{2n_1} \| |\text{RF}(f)|^2 \|_1 = 2^{2n_1} \|\text{RF}(f)\|_2^2 \lesssim \\
&\lesssim 2^{2n_1} \|f\|_2^2
\end{aligned}$$

We got the estimate

$$\sum_{I \in \mathcal{J}_d^{n_1}} |I| \lesssim 2^{2n_1} \|f\|_2^2.$$

Similarly,

$$\sum_{I \in \mathcal{J}_d^{n_2}} |I| \lesssim 2^{2n_2} \|g\|_2^2 \quad \text{and}$$

$$\sum_{I \in \mathcal{J}_d^{n_3}} |I| \lesssim 2^{n_3} \|h\|_1$$

which further implies for all $0 \leq \theta_1, \theta_2, \theta_3 \leq 1, \theta_1 + \theta_2 + \theta_3 = 1$

$$\sum_{I \in \mathcal{J}_d^{n_1, n_2, n_3}} |I| \lesssim 2^{2n_1\theta_1} \|f\|_2^{2\theta_1} 2^{2n_2\theta_2} \|g\|_2^{2\theta_2} 2^{n_3\theta_3} \|h\|_1^{\theta_3}.$$

Getting back to the trilinear form, we have

$$|\Lambda_{\mathcal{J}_d}(f, g, h)| \lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(1-2\theta_1)} 2^{-n_2(1-2\theta_2)} 2^{-n_3(1-\theta_3)} \|f\|_2^{2\theta_1} \|g\|_2^{2\theta_2} \|h\|_1^{\theta_3}.$$

Recalling the definition of Ω, \mathcal{J}_d , we have

$$2^{-n_1} \lesssim 2^{d/2}, \quad 2^{-n_2} \lesssim 2^{d/2}, \quad 2^{-n_3} \lesssim 2^{-100d}.$$

$$|\Lambda_{\mathcal{J}_d}(f, g, h)| \lesssim 2^{d(1-2\theta_1)} 2^{d(1-2\theta_2)} 2^{-100d(1-\theta_3)} |F|^{\theta_1} |G|^{\theta_2} |E|^{\theta_3},$$

as long as $1 - 2\theta_1 > 0, 1 - 2\theta_2 > 0, 1 - \theta_3 > 0$ and $\theta_1 + \theta_2 + \theta_3 = 1$. This is true for θ_1 and θ_2 strictly less than $1/2$. In particular if θ_1 and θ_2 are smaller than but arbitrarily close to $1/2$ and θ_3 close to 0 .

If we let $1/p = \theta_1, 1/q = \theta_2$ and $1/s' = \theta_3$ this becomes equivalent to

$$\Lambda_{\mathcal{J}_d} \lesssim 2^{-50d} |F|^{1/p} |G|^{1/q} |E|^{1/s'}$$

and thus $\Lambda(f, g, h)$ is of restricted weak type (p, q, s) for $p, q > 2$. So we get the whole range (p, q, s) , if $p, q > 2$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

The case $p = q = 2$ follows directly; it can't be obtained through interpolation methods.

CHAPTER 4

VECTOR VALUED EXTENSIONS FOR BILINEAR OPERATORS

4.1 Estimates for Localized *BHT*

Definition 26. If \mathbb{P} is a collection of tri-tiles and I is a dyadic interval, we denote by $\mathbb{P}(I)$ the tiles in \mathbb{P} whose spatial interval is contained in I :

$$\mathbb{P}(I) := \{P \in \mathbb{P} : I_P \subseteq I\}$$

Here we assume that F, G and H' are subsets of \mathbb{R} of finite measure and I_0 is a fixed dyadic interval. We are interested in finding estimates for the bilinear operator

$$BHT_{I_0}^{\text{loc}}(f, g)(x) := \sum_{P \in \mathbb{P}(I_0)} \frac{1}{|I_P|^{1/2}} \langle f \cdot \mathbf{1}_F, \phi_{P_1}^1 \rangle \langle g \cdot \mathbf{1}_G, \phi_{P_2}^2 \rangle \phi_{P_3}^3(x) \mathbf{1}_{H'}(x).$$

In doing so, we first study the associated trilinear form:

$$\Lambda_{BHT(I_0)}^{\text{loc}}(f, g, h) := \sum_{P \in \mathbb{P}(I_0)} \frac{1}{|I_P|^{1/2}} \langle f \cdot \mathbf{1}_F, \phi_{P_1}^1 \rangle \langle g \cdot \mathbf{1}_G, \phi_{P_2}^2 \rangle \langle h \cdot \mathbf{1}_{H'}, \phi_{P_3}^3 \rangle.$$

While this operator satisfies the same estimates as the Bilinear Hilbert Transform, the localization to the sets F, G and H' , and to the tiles in $\mathbb{P}(I_0)$ will bring some extra decay. First we prove a result in the "local L^2 case", when $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} \leq \frac{1}{2}$. This will illustrate the gist of our approach, most the technicalities.

Proposition 12 (The case $r_1, r_2, r' \geq 2$). *Let \mathbb{P} be a family of tri-tiles, I a dyadic interval and $H' \subset \mathbb{R}$ subset of finite measure. Then one can choose some positive numbers a_j so*

that

$$|\Lambda_{BHT;\mathbb{P}(I)}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'})| \lesssim \quad (4.1)$$

$$\lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F\right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G\right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'}\right)^{a_3} \|f \cdot \tilde{\chi}_I\|_{r_1} \|g \cdot \tilde{\chi}_I\|_{r_2} \|h \cdot \tilde{\chi}_I\|_{r'}. \quad (4.2)$$

Here $a_j = 1 - \frac{2}{r_j} - \epsilon > 0$, for some very small ϵ .

Proof. In this case we are proving restricted type estimates by applying directly Proposition 11: let E_1, E_2, E_3 be sets of finite measure, and $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E_2}, |h| \leq \mathbf{1}_{E_3}$. We have

$$\begin{aligned} \Lambda_{BHT}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'}) &\lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} f \cdot \mathbf{1}_F\right)^{\theta_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} g \cdot \mathbf{1}_G\right)^{\theta_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} h \cdot \mathbf{1}_{H'}\right)^{\theta_3} \\ &\quad \left(\text{energy } f \cdot \mathbf{1}_F\right)^{\frac{1-\theta_1}{2}} \left(\text{energy } g \cdot \mathbf{1}_G\right)^{\frac{1-\theta_2}{2}} \left(\text{energy } h \cdot \mathbf{1}_{H'}\right)^{\frac{1-\theta_3}{2}} \end{aligned}$$

Recall that the sizes can be estimated by

$$\widetilde{\text{size}}_{\mathbb{P}(I)} f \cdot \mathbf{1}_F \lesssim \sup_{P \in \mathbb{P}(I)} \frac{1}{|I_P|} \int \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_{I_P}^M dx$$

where M can be chosen to be as large as we wish. So if E_1 is supported away from I , the sizes will decay fast, giving the desired 4.1. Similarly for E_2 and E_3 . For this reason, we can assume that the sets E_1, E_2, E_3 are supported on $5I$ and then we will need to show only

$$|\Lambda_{BHT;\mathbb{P}(I)}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'})| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F\right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G\right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'}\right)^{a_3} \|f\|_{r_1} \|g\|_{r_2} \|h\|_{r'}.$$

The energies are bounded by L^2 norms, so we have

$$\Lambda_{BHT_{\text{loc}}}(f, g, h) \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F\right)^{\theta_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G\right)^{\theta_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'}\right)^{\theta_3} |E_1|^{\frac{1-\theta_1}{2}} |E_2|^{\frac{1-\theta_2}{2}} |E_3|^{\frac{1-\theta_3}{2}}.$$

This restricted type estimates are true in a very small neighborhood of $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'})$, and carefully interpolating the restricted type estimates one gets strong type estimates. \square

Now we treat the rest of the cases for which the admissible tuple $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'})$ is still in the Banach triangle:

$$0 < \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} < 1.$$

Proposition 13. *Let F, G and H' be as above and let $\mathbb{P}(I_0)$ be a family of tri-tiles localized to the dyadic interval I_0 . Then one can choose some positive numbers a^j so that*

$$|\Lambda_{BHT; \mathbb{P}(I_0)}^{loc}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'})| \lesssim \quad (4.3)$$

$$\lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F \right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G \right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'} \right)^{a_3} \|f \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h \cdot \tilde{\chi}_{I_0}\|_{r'}, \quad (4.4)$$

where $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1$, and $0 \leq a_j < 1$ for all $1 \leq j \leq 3$.

Proof. Start with E_1, E_2, E_3 sets of finite measure and $\tilde{\Omega}$ is an exceptional set defined as

$$\tilde{\Omega} := \{x : \mathcal{M}(\mathbf{1}_{E_1}) > C \frac{|E_1|}{|E_3|}\} \cup \{x : \mathcal{M}(\mathbf{1}_{E_2}) > C \frac{|E_2|}{|E_3|}\}.$$

Let $E'_3 := E_3 \setminus \tilde{\Omega}$. We want to prove that (4.3) holds for any functions f, g, h for which $|f| \leq \mathbf{1}_{E_1}$, $|g| \leq \mathbf{1}_{E_2}$, and $|h| \leq \mathbf{1}_{E'_3}$. For simplicity, we are making the following assumptions:

- i) the functions f, g and h are supported on $5I_0$;
- ii) for every tile $P \in \mathbb{P}(I_0)$, we have $1 + \frac{\text{dist}(I_P, \tilde{\Omega}^c)}{|I_P|} \sim 2^d$.

Then for every $P \in \mathbb{P}(I_0)$, we have

$$\frac{1}{|I_P|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_{I_P}^M dx \lesssim 2^d \frac{|E_1|}{|E_3|} \quad \text{and} \quad \frac{1}{|I_P|} \int_{\mathbb{R}} \mathbf{1}_{E_2} \cdot \mathbf{1}_G \cdot \tilde{\chi}_{I_P}^M dx \lesssim 2^d \frac{|E_2|}{|E_3|}.$$

This is important because now we can perform a stopping time which will allow us to estimate the ‘sizes’ of the functions $\mathbf{1}_{E_j}$.

We start with the largest value $2^{-l_1} \lesssim 2^d \frac{|E_1|}{|E_2|}$ and define \mathcal{J}_{l_1} to be the collection of maximal dyadic intervals I with the property that I contains some $I_P \in \mathbb{P}(I_0)$ which is not contained in any of the intervals previously selected, and I also has the property that

$$2^{-l_1-1} \leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \leq 2^{-l_1}.$$

The algorithm continues by decreasing 2^{-l_1} until all tiles in $\mathbb{P}(I_0)$ are exhausted. In this way, for any l_1 and any $I \in \mathcal{J}_{l_1}$ we have $\widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \sim 2^{-l_1}$. Similarly we construct the collections of dyadic intervals \mathcal{J}_{l_2} associated with the functions $\mathbf{1}_{E_2} \cdot \mathbf{1}_G$ as long as $2^{-l_2} \lesssim 2^d \frac{|E_2|}{|E_3|}$.

For the third component, the collections \mathcal{J}_{l_3} are non-empty as long as $2^{-n_3} \lesssim 2^{-\tilde{M}d}$, and in that case for any $I \in \mathcal{J}_{l_3}$, $\widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{H'} \cdot \mathbf{1}_{E'_3}) \sim 2^{-n_3}$. The extra decay in this case is due to the fact that E'_3 is actually supported on $\tilde{\Omega}^c$.

Given l_1, l_2, l_3 as above, we denote $\mathcal{J}^{l_1, l_2, l_3} := \mathcal{J}_{l_1} \cap \mathcal{J}_{l_2} \cap \mathcal{J}_{l_3}$. This is also going to be a collection of dyadic intervals, and moreover $\mathbb{P}(I_0) = \bigcup_{l_1, l_2, l_3} \bigcup_{I \in \mathcal{J}^{l_1, l_2, l_3}} \mathbb{P}(I)$. Thus

$$\Lambda_{BHT; \mathbb{P}(I_0)}^{\text{loc}}(f, g, h) = \sum_{l_1, l_2, l_3} \sum_{I \in \mathcal{J}^{l_1, l_2, l_3}} \Lambda_{BHT; \mathbb{P}(I)}^{\text{loc}}(f, g, h). \quad (4.5)$$

Every $\Lambda_{BHT; \mathbb{P}(I)}^{\text{loc}}(f, g, h)$ will be estimated with the use of Proposition 11:

$$\begin{aligned} \Lambda_{BHT; \mathbb{P}(I)}^{\text{loc}}(f, g, h) &\lesssim \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F)^{\theta_1} \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_2} \cdot \mathbf{1}_G)^{\theta_2} \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E'_3} \cdot \mathbf{1}_{H'})^{\theta_3} \\ &\quad \text{energy}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F)^{1-\theta_1} \text{energy}_{\mathbb{P}(I)}(\mathbf{1}_{E_2} \cdot \mathbf{1}_G)^{1-\theta_2} \text{energy}_{\mathbb{P}(I)}(\mathbf{1}_{E'_3} \cdot \mathbf{1}_{H'})^{1-\theta_3}. \end{aligned}$$

Lemmas 3 and 4 will help with the sizes and energies, but in the case of the energy, one can do even better:

$$\text{energy}_{\mathbb{P}(I)}(f) \sim \left(\sum_{T \in \mathbb{P}(I)} \sum_{T \in \mathbb{T}} \sum_{P \in T} |\langle f, \phi_P \rangle|^2 \right)^{1/2} \lesssim \|f \cdot \tilde{\chi}_I\|_2 \quad (4.6)$$

This is because for all ϕ_P , the information is mainly concentrated on the interval I . More exactly, if \mathbb{T} is a family of disjoint trees that have disjoint phase intervals (that is, the intervals I_T are disjoint), and f is supported on $(5I)^c$, then we have

$$\begin{aligned} & \left(\sum_{T \in \mathbb{T}} \sum_{\substack{P \in \mathbb{T} \\ I_P \subseteq I}} |\langle f, \phi_P \rangle|^2 \right)^{1/2} \leq \sum_{T \in \mathbb{T}} \sum_{\substack{P \in \mathbb{T} \\ I_P \subseteq I}} |\langle f, \phi_P \rangle| \leq \\ & \leq \|f\|_2 \sum_{T \in \mathbb{T}} \sum_{\substack{P \in \mathbb{T} \\ I_P \subseteq I}} \left(\frac{\text{dist}(I_P, 5I^c)}{|I_P|} \right)^{-100} \lesssim \|f\|_2 \cdot \sum_{I_P \subseteq I} \left(\frac{\text{dist}(I_P, 5I^c)}{|I_P|} \right)^{-100} \lesssim \\ & \lesssim \|f\|_2 \cdot \sum_{k \geq 0} \sum_{\substack{|I_P| = 2^{-k}|I| \\ I_P \subseteq I}} \left(\frac{\text{dist}(I_P, 5I^c)}{|I|} \cdot \frac{|I|}{|I_P|} \right)^{-100} \lesssim \|f\|_2 \cdot \sum_{k \geq 0} 2^{-50k} \left(\frac{\text{dist}(I, 5I^c)^{50}}{|I|} \right). \end{aligned}$$

For the particular function $\mathbf{1}_{E_1} \cdot \mathbf{1}_F$ and some interval $I \in \mathcal{I}^{l_1, l_2, l_3}$ we have

$$\text{energy}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \lesssim \left(\int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \right)^{1/2} \lesssim (2^{-l_1} |I|)^{1/2} \lesssim (\text{size}_{\mathbb{P}(I_0)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \cdot |I|)^{1/2}.$$

In this way 4.5 becomes

$$\Lambda_{BHT; \mathbb{P}(I_0)}^{\text{loc}}(f, g, h) \lesssim \sum_{l_1, l_2, l_3} \sum_{I \in \mathcal{I}^{l_1, l_2, l_3}} \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F)^{\theta_1} \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_2} \cdot \mathbf{1}_G)^{\theta_2} \widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E'_3} \cdot \mathbf{1}_{H'})^{\theta_3}. \quad (4.7)$$

$$\left(\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \right)^{\frac{1-\theta_1}{2}} \left(\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_2} \cdot \mathbf{1}_G \cdot \tilde{\chi}_I^M dx \right)^{\frac{1-\theta_2}{2}} \left(\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E'_3} \cdot \mathbf{1}_{H'} \cdot \tilde{\chi}_I^M dx \right)^{\frac{1-\theta_3}{2}} \cdot |I| \lesssim \quad (4.8)$$

$$\lesssim \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_F)^{\frac{1+\theta_1}{2} - \frac{1}{r_1}} \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_G)^{\frac{1+\theta_2}{2} - \frac{1}{r_2}} \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_{H'})^{\frac{1+\theta_3}{2} - \frac{1}{r'} - \epsilon}. \quad (4.9)$$

$$\cdot \sum_{l_1, l_2, l_3} \sum_{I \in \mathcal{I}^{l_1, l_2, l_3}} 2^{-\frac{l_1}{r_1}} 2^{-\frac{l_2}{r_2}} 2^{-l_3(\frac{1}{r'} + \epsilon)} |I|, \quad (4.10)$$

as long as

$$\frac{1+\theta_1}{2} - \frac{1}{r_1} > 0, \quad \frac{1+\theta_2}{2} - \frac{1}{r_2} > 0, \quad \frac{1+\theta_3}{2} - \frac{1}{r'} > 0. \quad (4.11)$$

The quantity

$$\widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_F)^{\frac{1+\theta_1}{2} - \frac{1}{r_1}} \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_G)^{\frac{1+\theta_2}{2} - \frac{1}{r_2}} \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_{H'})^{\frac{1+\theta_3}{2} - \frac{1}{r'} - \epsilon}$$

is going to represent the operatorial norm associated with the trilinear form

$$\Lambda_{BHT;\mathbb{P}(I_0)}^{\text{loc}}.$$

$\sum_{I \in \mathcal{J}^{l_1, l_2, l_3}} |I|$ can be estimated in three different ways; for example,

$$\sum_{I \in \mathcal{J}^{l_1, l_2, l_3}} |I| \leq \sum_{I \in \mathcal{J}_{l_1}} |I| = \left\| \sum_{I \in \mathcal{J}_{l_1}} \mathbf{1}_I \right\|_{1, \infty} \lesssim \left\| \sum_{I \in \mathcal{J}_{l_1}} 2^{l_1} \mathcal{M} \mathbf{1}_{E_1} \cdot \mathbf{1}_I \right\|_{1, \infty} \lesssim 2^{n_1} |E_1|.$$

For this reason, whenever $0 \leq \alpha_j \leq 1$, with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we have

$$\sum_{I \in \mathcal{J}^{l_1, l_2, l_3}} |I| \lesssim \left(2^{l_1} |E_1|\right)^{\alpha_1} \left(2^{l_2} |E_2|\right)^{\alpha_2} \left(2^{l_3} |E_3|\right)^{\alpha_3}.$$

This will imply that

$$\begin{aligned} & \sum_{l_1, l_2, l_3} \sum_{I \in \mathcal{J}^{l_1, l_2, l_3}} 2^{-\frac{l_1}{r_1}} 2^{-\frac{l_2}{r_2}} 2^{-l_3(\frac{1}{r'} + \epsilon)} |I| \lesssim \\ & \lesssim \sum_{l_1, l_2, l_3} 2^{-l_1(\frac{1}{r_1} - \alpha_1)} 2^{-l_2(\frac{1}{r_2} - \alpha_2)} 2^{-l_3(\frac{1}{r'} + \epsilon - \alpha_1)} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} \lesssim \\ & \lesssim \left(2^d \frac{|E_1|}{|E_3|}\right)^{\frac{1}{r_1} - \alpha_1} \left(2^d \frac{|E_2|}{|E_3|}\right)^{\frac{1}{r_2} - \alpha_2} \left(2^{-\tilde{M}d}\right)^{(\frac{1}{r'} + \epsilon - \alpha_3)} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} \lesssim \\ & \lesssim 2^{-100d} |E_1|^{1/r_1} |E_2|^{1/r_2} |E_3|^{1/r'} \end{aligned}$$

The series above will converge because $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} + \epsilon > 1$.

Hence, by interpolation, and after removing the assumption i), we get

$$\begin{aligned} & |\Lambda_{BHT;\mathbb{P}(I_0)}^{\text{loc}}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'})| \lesssim \\ & \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F\right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G\right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'}\right)^{a_3} \|f \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h \cdot \tilde{\chi}_{I_0}\|_{r'}, \end{aligned}$$

where a_1, a_2 and a_3 can be described as

$$a_1 = \frac{1 + \theta_1}{2} - \frac{1}{r_1}, \quad a_2 = \frac{1 + \theta_2}{2} - \frac{1}{r_2}, \quad a_3 = \frac{1 + \theta_3}{2} - \frac{1}{r'}$$

for some $0 \leq \theta_1, \theta_2, \theta_3 < 1$, with $\theta_1 + \theta_2 + \theta_3 = 1$ than will be chosen later. \square

Remark 8. If f was so that $\text{supp } f \subseteq 2^{k+1}I \setminus 2^kI$, then

$$(\text{size}_{\mathbb{P}(I)}(f \cdot \mathbf{1}_F))^{\theta_1} \lesssim 2^{-10kM\theta_1} \text{size}_{\mathbb{P}(I)}(f \cdot \mathbf{1}_F)^{\theta_1 - \epsilon}.$$

Because of the extra decay 2^{-10kM} we can assume that the functions f, g and h are primarily supported on $5I$.

Corollary 2. Let θ_j be so that $\frac{1}{r_1} \leq \frac{1 + \theta_1}{2}$, $\frac{1}{r_2} \leq \frac{1 + \theta_2}{2}$, and $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$\|BHT_{\text{loc}}(f, g)\|_1 \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F\right)^{\frac{1+\theta_1}{2} - \frac{1}{r_1}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G\right)^{\frac{1+\theta_2}{2} - \frac{1}{r_2}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'}\right)^{\frac{1+\theta_3}{2}} \|f \cdot \tilde{\chi}_I\|_{r_1} \|g \cdot \tilde{\chi}_I\|_{r_2}.$$

Proof. A careful examination of 4.7 shows that one can choose any triple $(\beta_1, \beta_2, \beta_3)$ with $\beta_1 + \beta_2 + \beta_3 = 1$, even with $\beta_3 \leq 0$, in the place of $(1/r_1, 1/r_2, 1/r')$.

In this case we get

$$|\Lambda_{BHT; \mathbb{P}(I_0)}^{\text{loc}}(f, g, h)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F\right)^{\frac{1+\theta_1}{2} - \beta_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G\right)^{\frac{1+\theta_2}{2} - \beta_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'}\right)^{\frac{1+\theta_3}{2} - \epsilon}.$$

$$|E_1|^{\beta_1} |E_2|^{\beta_2} |E_3|^{\beta_3}$$

The restrictions are that $\beta_j < \frac{1 + \theta_j}{2}$, which works well for very small or negative values of β_3 . Considering tuples $(\beta_1, \beta_2, \beta_3)$ that lie in a small open neighborhood of $\left(\frac{1}{r_1}, \frac{1}{r_2}, 0\right)$, we get the conclusion. In this case, the interpolation doesn't yield estimates for the trilinear form $\Lambda_{BHT; \mathbb{P}(I_0)}^{\text{loc}}$, but for the L^1 norm of the operator. \square

4.2 Vector Valued BHT

One of the main results in this thesis is a vector-valued extension of BHT , for any triple $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'})$ with

$$0 \leq \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} < 1, \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1.$$

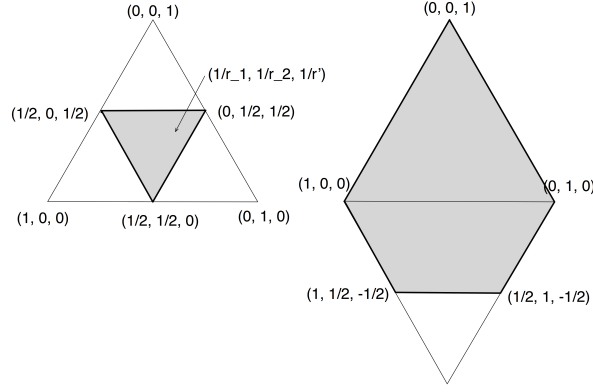


Figure 4.1: Range for vector valued *BHT* when $1/r_1, 1/r_2, 1/r' \leq 1/2$

In particular, when $p, q \geq 2$, vector-valued estimates are available for any r_1, r_2 and r .

Theorem 20. For any r_1, r_2, r' so that $0 \leq \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} < 1$, and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1$, we have

$$\left\| \left(\sum_{k=1}^N |BHT(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_{k=1}^N |f_k|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_{k=1}^N |g_k|^{r_2} \right)^{1/r_2} \right\|_q.$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and $1 < p, q \leq \infty, 2/3 < s < \infty$, and the exponents $(\frac{1}{p}, \frac{1}{q}, \frac{1}{s})$ satisfy the following:

- i) if $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} \leq \frac{1}{2}$, then we don't have any further restrictions, so the range of exponents is the one of the *BHT* operator: $\mathcal{D}_{r_1, r_2, r'} = \text{Range}(BHT)$.
- ii) if $\frac{1}{r_2}, \frac{1}{r'} \leq \frac{1}{2}, \frac{1}{r_1} > \frac{1}{2}$, then $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s}\right)$ are in $\mathcal{D}_{r_1, r_2, r'}$, which is the convex hull of

$$\text{Co} \left((0, 0, 1), (1, 0, 0), \left(1, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{r_1}, \frac{3}{2} - \frac{1}{r_1}, -\frac{1}{2}\right), \left(0, \frac{3}{2} - \frac{1}{r_1}, \frac{1}{r_1} - \frac{1}{2}\right) \right).$$

- iii) if $\frac{1}{r_1}, \frac{1}{r'} \leq \frac{1}{2}, \frac{1}{r_2} > \frac{1}{2}$, then the range of exponents is similar to the one in ii), with the role of r_1 and r_2 switched.

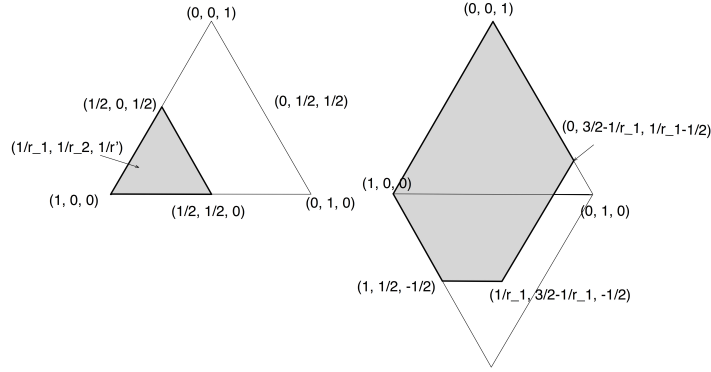


Figure 4.2: Range for vector valued BHT when $1/r_1 > 1/2$

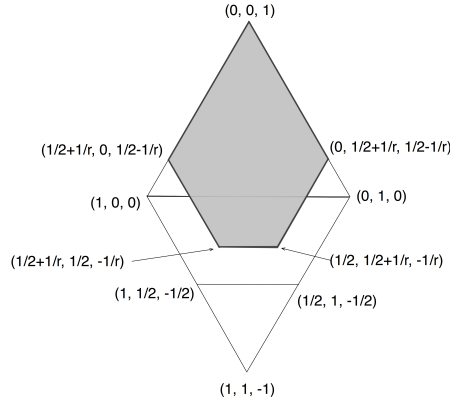


Figure 4.3: Range for vector valued BHT when $1/r' > 1/2$

iv) if $\frac{1}{r_1}, \frac{1}{r_2} \leq \frac{1}{2}, \frac{1}{r'} > \frac{1}{2}$, then $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'}\right)$ are in $\mathcal{D}_{r_1, r_2, r'} :=$ the convex hull of

$$\text{Co}\left((0, 0, 1), \left(\frac{1}{2} + \frac{1}{r}, 0, -\frac{1}{r}\right), \left(\frac{1}{2} + \frac{1}{r}, \frac{1}{2}, -\frac{1}{r}\right), \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{r}, -\frac{1}{r}\right), \left(0, \frac{1}{2} + \frac{1}{r}, \frac{1}{2} - \frac{1}{r}\right)\right).$$

Remark 9. 1. Estimates still hold when one of p or q equals ∞ . In general, for the boundary points we can only hope to get weak type estimates.

2. Note that whenever (p, q, s') are so that $0 \leq \frac{1}{p}, \frac{1}{q} \leq \frac{1}{2}$ and $s \neq \infty$, vector valued estimates exist for any tuple (r_1, r_2, r) .

The method of the proof is based on *generalized restricted type* interpolation. For the Banach-valued multilinear operator BHT , the trilinear form is given by

$$\Lambda_{BHT}(f, g, h) := \langle BHT(f, g), h \rangle = \sum_{k=1}^N \Lambda_{BHT; \mathbb{P}_k}(f_k, g_k, h_k) \quad (4.12)$$

where $f = \{f_k\}_k \in L^p(l^{r_1})$, $g = \{g_k\}_k \in L^q(l^{r_2})$ and $h = \{h_k\}_k \in L^{s'}(l^{r'})$.

For each $BHT(f_k, g_k)$ we could use a different collection \mathbb{P}_k of tiles for the discretized model operator:

$$BHT_{\mathbb{P}_k}(f_k, g_k)(x) = \sum_{P \in \mathbb{P}_k} \frac{1}{|I_P|^{1/2}} \langle f_k, \phi_1^1 \rangle \langle g_k, \phi_2^2 \rangle \phi_3^3(x).$$

Remark 10. *The results we are proving are true even if we use different model operators for $BHT(f_k, g_k)$; so theorem 20 allows for a slight generalization:*

$$\left\| \left(\sum_{k=1}^N |\widetilde{BHT}_k(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_{k=1}^N |f_k|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_{k=1}^N |g_k|^{r_2} \right)^{1/r_2} \right\|_q$$

The operators \widetilde{BHT}_k have properties similar to the Bilinear Hilbert Transform, but they need not be exactly the same.

As mentioned before, we will prove generalized restricted type estimates for the vector-valued $\Lambda_{BHT}(f, g, h)$. Let F, G and H be sets of finite measure, with $|H| = 1$. In what follows, we will construct a major subset $H' \subseteq H$ and show

$$\left| \sum_{k=1}^N \Lambda_{BHT; \mathbb{P}}(f_k, g_k, h_k) \right| \lesssim |F|^{\alpha_1} |G|^{\alpha_2} |H|^{\alpha_3} \quad (4.13)$$

whenever $\left(\sum_k |f_k|^{r_2} \right)^{1/r_1} \leq \mathbf{1}_F$, $\left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \leq \mathbf{1}_G$ and $\left(\sum_k |h_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_{H'}$. The exceptional set is defined as

$$\Omega := \{x : \mathcal{M}(\mathbf{1}_F) > C|F|\} \cup \{x : \mathcal{M}(\mathbf{1}_G) > C|G|\}.$$

Because of the $L^1 \rightarrow L^{1,\infty}$ boundedness of the maximal operator, for C large enough $|\Omega| < 1$.

We partition the collection of tri-tiles according to the scaled distance from the exceptional set

$$\mathbb{P}^d = \{P \in \mathbb{P}_k : 1 + \frac{\text{dist}(I_P, \Omega^c)}{|I_P|} \sim 2^d\}$$

and will prove estimates equivalent to (4.13) for the family \mathbb{P}^d , with an extra 2^{-10d} decay:

$$|\sum_{k=1}^N \Lambda_{BHT; \mathbb{P}^d}(f_k, g_k, h_k)| \lesssim 2^{-10d} |F|^{1/p} |G|^{1/q} |H|^{1/s'}. \quad (4.14)$$

We suppress the d -dependency for the moment.

Now we construct a collection $\{\mathcal{I}^{n_1}\}_{n_1 \geq \bar{n}_1}$ of relevant dyadic intervals according to the “concentration” of $\mathbf{1}_F$ among the phase intervals I_P :

- start with \bar{n}_1 so that $2^{-\bar{n}_1} \sim 2^d |F|$ and let $\mathbb{P}'_{\bar{n}_1-1} = \mathbb{P}$ (\mathbb{P}'_{n_1} will play the role of *Stock*)
- define $\mathcal{I}_1^{\bar{n}_1}$ to be the collection of maximal dyadic intervals I with the property that there exists at least one tile $P \in \mathbb{P}'_{\bar{n}_1}$ with $I_P \subseteq I$ and

$$\frac{1}{|I|} \int \mathbf{1}_F \cdot \tilde{\chi}_I dx \sim 2^{-\bar{n}_1}. \quad (4.15)$$

- for every such interval I , let $\mathbb{P}_{\bar{n}_1}(I)$ be the collection of tiles $P \in \mathbb{P}'_{\bar{n}_1}$ with the property that $I_P \subseteq I$
- set $\mathbb{P}'_{\bar{n}_1} = \mathbb{P} \setminus \bigcup_{I \in \mathcal{I}_1^{\bar{n}_1}} \mathbb{P}_{\bar{n}_1}(I)$
- for all $n_1 \geq \bar{n}_1$, $\mathcal{I}_1^{n_1}$ will denote the collection of maximal dyadic intervals which contain a phase interval I_P for some $P \in \mathbb{P}'_{n_1+1}$ (which was not selected previously) and so that

$$2^{-n_1-1} \leq \frac{1}{|I|} \int \mathbf{1}_F \cdot \tilde{\chi}_I^M dx < 2^{-n_1}.$$

- as before, $\mathbb{P}_{n_1}(I) := \{P \in \mathbb{P}'_{n_1} : I_P \subseteq I\}$
- set $\mathbb{P}'_{n_1} = \mathbb{P}_{n_1+1} \setminus \bigcup_{I \in \mathcal{J}_1^{n_1}} \mathbb{P}_{n_1}(I)$
- note that we always have $2^{-n_1} \lesssim 2^d |F|$.

For d sufficiently large, the intervals I_P for $P \in \mathbb{P}^d$ are going to be essentially disjoint and the intervals $I \in \mathcal{J}_1^{n_1}$ can be selected in an easier way; but this is not the case for example when $d = 0$, which corresponds to $I_P \cap \Omega \neq \emptyset$. However, for every n_1 , the intervals in $\mathcal{J}_1^{n_1}$ are going to be disjoint and this is all that is going to be used later in the proof.

Similarly, $\mathcal{J}_2^{n_2}$ denotes the collection of maximal dyadic intervals I containing at least some $I_P \subseteq I$ for some $P \in \mathbb{P}^d$, and

$$\frac{1}{|I|} \int \mathbf{1}_G \cdot \tilde{\chi}_I^M dx \sim 2^{-n_2} \lesssim 2^d |G|.$$

For $\mathbf{1}_{H'}$, $\mathcal{J}_3^{n_3}$ = collection of maximal dyadic intervals I containing at least some I_P for some $P \in \mathbb{P}^d$ and so that

$$\frac{1}{|I|} \int \mathbf{1}_{H'} \cdot \tilde{\chi}_I^M dx \sim 2^{-n_3} \lesssim 2^{-Md}.$$

We denote $\mathcal{J}^{n_1, n_2, n_3} := \mathcal{J}_1^{n_1} \cap \mathcal{J}_2^{n_2} \cap \mathcal{J}_3^{n_3}$.

Next we further partition \mathbb{P}^d as $\mathbb{P}^d = \bigcup_{n_1, n_2, n_3} \bigcup_{I \in \mathcal{J}^{n_1, n_2, n_3}} \mathbb{P}(I)$.

For $I \in \mathcal{J}_1^{n_1}$, we have $\widetilde{\text{size}}_{\mathbb{P}_{n_1}(I)} \mathbf{1}_F \sim 2^{-n_1}$. When we intersect different intervals in $\mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}$ and $\mathcal{J}_3^{n_3}$ all we can say is that $\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \lesssim 2^{-n_1}$. This fact is the technical obstruction in obtaining vector valued *BHT* estimates for any p, q, s in the range of *BHT*.

In a similar way, the relation $\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \sim 2^{-n_1}$ for $I \in \mathcal{I}_1^{n_1}$ becomes, for an interval $I' \in \mathcal{I}_1^{n_1} \cap \mathcal{I}_2^{n_2} \cap \mathcal{I}_3^{n_3}$

$$\frac{1}{|I'|} \int_{\mathbb{R}} \mathbf{1}_F \tilde{\chi}_{I'}^M dx \lesssim 2^{-n_1}.$$

The trilinear form in (4.14) becomes

$$\sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{I}_1^{n_1}, \mathcal{I}_2^{n_2}, \mathcal{I}_3^{n_3}} \sum_k \Lambda_{BHT; \mathbb{P}(I)}(f_k, g_k, h_k).$$

Note that the functions f_k are supported on F , the g_k s on G and the h_k s on H' .

We can apply the localization proposition (13) to get

$$|\Lambda_{BHT; \mathbb{P}(I)}(f_k, g_k, h_k)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{b_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{b_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{b_3} \cdot \\ \|f_k \cdot \tilde{\chi}_I\|_{r_1} \|g_k \cdot \tilde{\chi}_I\|_{r_2} \|h_k \cdot \tilde{\chi}_I\|_{r'}.$$

where $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1$, and $0 \leq b_j \leq a_j$.

Recall that the a_j were given by

$$a_1 = \frac{1 + \theta_1}{2} - \frac{1}{r_1}, \quad a_2 = \frac{1 + \theta_2}{2} - \frac{1}{r_2}, \quad a_3 = \frac{1 + \theta_3}{2} - \frac{1}{r'}$$

where the only conditions we have on θ_1, θ_2 and θ_3 are that $\theta_1 + \theta_2 + \theta_3 = 1$ and $a_j > 0$.

Using Hölder's inequality, the initial tri-linear form can be estimated by

$$\sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{I}_1^{n_1}, \mathcal{I}_2^{n_2}, \mathcal{I}_3^{n_3}} \sum_k |\Lambda_{BHT; \mathbb{P}(I)}(f_k, g_k, h_k)| \lesssim \\ \lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{I}_1^{n_1}, \mathcal{I}_2^{n_2}, \mathcal{I}_3^{n_3}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{a_3} \cdot \\ \|(\sum_k |f_k|^{r_1})^{1/r_1} \tilde{\chi}_I\|_{r_1} \|(\sum_k |g_k|^{r_2})^{1/r_2} \tilde{\chi}_I\|_{r_2} \|(\sum_k |h_k|^{r'})^{1/r'} \tilde{\chi}_I\|_{r'} \lesssim$$

$$\begin{aligned}
&\lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{a_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{a_2} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{a_3} \cdot \\
&\frac{\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{r_1}}{|I|^{1/r_1}} \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_I\|_{r_2}}{|I|^{1/r_2}} \frac{\|\mathbf{1}_{H'} \cdot \tilde{\chi}_I\|_{r'}}{|I|^{1/r'}} \cdot |I| \lesssim \\
&\lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} 2^{-n_1/p} 2^{-n_2/q} 2^{-n_3(a_3 + \frac{1}{r'})} \cdot |I|
\end{aligned}$$

In the last inequality we need to assume $\frac{1}{p} \leq a_1 + \frac{1}{r_1} = \frac{1 + \theta_1}{2}$ and similarly $\leq \frac{1 + \theta_2}{r'}$. We will be summing $|I|$ when $I \in \mathcal{J}^{n_1, n_2, n_3}$. Note

$$\begin{aligned}
\sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} |I| &\leq \sum_{I \in \mathcal{J}_1^{n_1}} |I| = \left\| \sum_{I \in \mathcal{J}_1^{n_1}} \mathbf{1}_I \right\|_{1, \infty} \lesssim \\
&\lesssim \left\| \sum_{I \in \mathcal{J}_1^{n_1}} 2^{n_1} (\mathcal{M} \mathbf{1}_F) \cdot \mathbf{1}_I \right\|_{1, \infty} \lesssim 2^{n_1} |F|.
\end{aligned}$$

Similarly,

$$\sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} |I| \lesssim 2^{n_2} |G|, 2^{n_3} |H|$$

and interpolating these three inequalities we get

$$\sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} |I| \lesssim (2^{n_1} |F|)^{\gamma_1} (2^{n_2} |G|)^{\gamma_2} (2^{n_3} |H|)^{\gamma_3}$$

where $0 \leq \gamma_j \leq 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Finally,

$$\begin{aligned}
&\left| \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} \sum_k \Lambda_{BHT; \mathbb{P}_k(I)}(f_k, g_k, h_k) \right| \lesssim \\
&\sum_{n_1, n_2, n_3} 2^{-n_1/p} 2^{-n_2/q} 2^{-n_3 \frac{1+\theta_3}{2}} (2^{n_1} |F|)^{\gamma_1} (2^{n_2} |G|)^{\gamma_2} (2^{n_3} |H|)^{\gamma_3} \lesssim \\
&\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(1/p - \gamma_1)} 2^{-n_2(1/q - \gamma_2)} 2^{-n_3(\frac{1+\theta_3}{2} - \gamma_3)} |F|^{\gamma_1} |G|^{\gamma_2}
\end{aligned}$$

The series converges if we can pick γ_j so that

$$\frac{1}{p} > \gamma_1, \quad \frac{1}{q} > \gamma_2 \quad \text{and} \quad \frac{1 + \theta_3}{2} > \gamma_3.$$

This will be possible as long as

$$\frac{1}{p} + \frac{1}{q} + \frac{1 + \theta_3}{2} > 1. \tag{4.16}$$

In case the above conditions are satisfied, we get generalized restricted type estimate

$$|\sum_k \Lambda_k(f_k, g_k, h_k)| \lesssim |F|^{1/p} |G|^{1/q}.$$

There are four distinct cases:

- i) $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'} \leq \frac{1}{2}$. In this case, if we pick $\theta_1 = \theta_2 \sim 0$ and $\theta_3 \sim 1$, all the conditions hold and the range is going to be the convex hull of the points

$$(0, 0, 1), (1, 0, 0), \left(1, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, 1, -\frac{1}{2}\right), (0, 1, 0).$$

That is, we get the same range as that for the *BHT* operator: $p, q \geq 1, s > \frac{2}{3}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

- ii) $\frac{1}{r_2}, \frac{1}{r'} \leq \frac{1}{2}$ and $\frac{1}{r_1} > \frac{1}{2}$. For the condition $\frac{1+\theta_1}{2} - \frac{1}{r_1} > 0$ to hold, we have to choose $\theta_1 > \frac{2}{r_1} - 1$ and this will imply that the range of the operator described as a region in the hyperplane $\beta_1 + \beta_2 + \beta_3 = 1$ is the convex hull of the points

$$(0, 0, 1), (1, 0, 0), \left(1, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{r_1}, \frac{3}{2} - \frac{1}{r_1}, -\frac{1}{2}\right), \left(0, \frac{3}{2} - \frac{1}{r_1}, \frac{1}{r_1} - \frac{1}{2}\right).$$

- iii) $\frac{1}{r_1}, \frac{1}{r'} \leq \frac{1}{2}$, and $\frac{1}{r_2} > \frac{1}{2}$. Analogous to the previous case, the range of the operator is the convex hull of

$$(0, 0, 1), (0, 1, 0), \left(1, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{3}{2} - \frac{1}{r_2}, \frac{1}{r_2}, -\frac{1}{2}\right), \left(\frac{3}{2} - \frac{1}{r_2}, 0, \frac{1}{r_2} - \frac{1}{2}\right).$$

- iv) $\frac{1}{r_1}, \frac{1}{r_2} \leq \frac{1}{2}$, $\frac{1}{r'} > \frac{1}{2}$. The range is the convex hull of

$$(0, 0, 1), \left(\frac{1}{2} + \frac{1}{r}, 0, \frac{1}{2} - \frac{1}{r}\right), \left(\frac{1}{2} + \frac{1}{r}, \frac{1}{2}, -\frac{1}{r}\right), \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{r}, -\frac{1}{r}\right), \left(0, \frac{1}{2} + \frac{1}{r}, \frac{1}{2} - \frac{1}{r}\right).$$

4.2.1 The cases $r = 1$ or $r_i = \infty$

The proof is similar to the one in the previous section 4.2. First consider the case $r = 1$. Because the dual space of l^1 is l^∞ , the sequences of functions will satisfy

$$\left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \leq \mathbf{1}_F, \quad \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \leq \mathbf{1}_G, \quad \sup_k |h_k| \leq \mathbf{1}_{H'}.$$

All the details are identical to the case $r > 1$; the restrictions are given by only two inequalities:

$$\frac{1 + \theta_1}{2} \geq \frac{1}{r_1}, \quad \frac{1 + \theta_2}{2} \geq \frac{1}{r_2}.$$

In the case $r_1 = r_2 = 2$, $r = 1$, these are automatically satisfied and the range we have for p, q , and s is the same as that for the BHT operator.

When $r_1 = \infty$, we use the fact that the first adjoint of BHT is a bilinear operator of the same kind, which is bounded from $L^r \times L^{r'} \rightarrow L^1$; more precisely,

$$\Lambda_{BHT}(f_k, g_k, h_k) = \int_{\mathbb{R}} BHT(f_k, g_k)(x) \cdot h_k(x) dx = \int_{\mathbb{R}} f_k(x) \cdot BHT^{*,1}(g_k, h_k)(x) dx.$$

In proving the boundedness of vector valued BHT via interpolation, we assume $\sup_k |f_k| \leq \mathbf{1}_F$, $\left(\sum_k |g_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_G$ and $\left(\sum_k |h_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_{H'}$. Then

$$\begin{aligned} |\Lambda_{BHT; \mathbb{P}(I)}(f_k, g_k, h_k)| &\leq \|BHT_{\mathbb{P}(I)}^{*,1}(g_k \cdot \mathbf{1}_G, h_k \cdot \mathbf{1}_{H'}) \cdot \mathbf{1}_F\|_1 \lesssim \\ &\lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{r}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{r'}} \cdot \|g_k \cdot \tilde{\chi}_I\|_r \|h_k \cdot \tilde{\chi}_I\|_{r'}. \end{aligned}$$

The rest follows as before. Note that in the case $(\infty, 2, 2)$ we have no constraints on p, q , and s except for those coming from the original BHT operator itself.

4.3 Iterated l^p spaces estimates for BHT

Previously, we proved that for any tuple (r_1, r_2, r) with $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r'}$, $1 \leq r < \infty$, and $1 < r_1, r_2 \leq \infty$, we have

$$BHT : L^p(l^{r_1}) \times L^q(l^{r_2}) \rightarrow L^s(l^r)$$

whenever p, q, r are in a certain range \mathcal{D}_{r,r_1,r_2} which can be described in a precise manner. In this section, we will describe a way to obtain a localized version of this result, as well as a vector valued extension for BHT when the vector spaces are iterated l^p spaces. We state the theorem in its generality, but will illustrate mainly the case of two iterated spaces $l^s(l^r)$ in order to simplify the notation.

Theorem 21. *Let M be an arbitrary positive integer, and consider tuples (s_{k1}, s_{k2}, s_k) with the property that*

$$\frac{1}{s_{k1}} + \frac{1}{s_{k2}} = \frac{1}{s_k}, \quad 1 < s_{k1}, s_{k2} \leq \infty, 1 \leq s_k < \infty$$

and $(s_{k1}, s_{k2}, s_k) \in \mathcal{D}_{(s_{(k-1)1}, s_{(k-1)2}, s_{(k-1)})}$. Then

$$\left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |BHT(f_{k_1 \dots k_M}, g_{k_1 \dots k_M})|^{s_1} \right)^{s_2/s_1} \dots \right)^{1/s_M} \right\|_t \leq \quad (4.17)$$

$$\leq C \left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |f_{k_1 \dots k_M}|^{s_{11}} \right)^{s_{21}/s_{11}} \dots \right)^{1/s_{M1}} \right\|_p \left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |g_{k_1 \dots k_M}|^{s_{12}} \right)^{s_{22}/s_{12}} \dots \right)^{1/s_{M2}} \right\|_q \quad (4.18)$$

whenever $(p, q, t) \in \mathcal{D}_{(s_{M1}, s_{M2}, s_M)} \cap \text{Range}(BHT)$.

Moreover, if all the s_{kj} satisfy $0 \leq \frac{1}{s_{k1}}, \frac{1}{s_{k2}}, \frac{1}{s_k} \leq \frac{1}{2}$, then the only constraint on p, q, t is that $(p, q, t) \in \text{Range}(BHT)$.

In proving this, one uses a localized version of the above result, which can be inductively proved.

Lemma 6. *Under the same hypotheses as in Theorem 21, given a dyadic interval I_0 and measurable sets of finite measure F, G, H' , we have*

$$\left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |BHT_{\mathbb{P}(I_0)}(f_{k_1 \dots k_M} \cdot \mathbf{1}_F, g_{k_1 \dots k_M} \cdot \mathbf{1}_G) \cdot \mathbf{1}_{H'}|^{s_1} \right)^{s_2/s_1} \dots \right)^{1/s_M} \right\|_t \leq \quad (4.19)$$

$$\leq \tilde{C} \left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |f_{k_1 \dots k_M}|^{s_{11}} \right)^{s_{21}/s_{11}} \dots \right)^{1/s_{M1}} \right\|_p \cdot \tilde{\chi}_{I_0} \|_p \quad (4.20)$$

$$\cdot \left\| \left(\sum_{k_M} \dots \left(\sum_{k_1} |g_{k_1 \dots k_M}|^{s_{12}} \right)^{s_{22}/s_{12}} \dots \right)^{1/s_{M2}} \right\|_q \cdot \tilde{\chi}_{I_0} \|_q \quad (4.21)$$

where p, q, t are in the Banach range, and the constant \tilde{C} is given by

$$\left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{q}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{t}}. \quad (4.22)$$

Remark 11. *The localization result is formulated for (p, q, t) in the Banach range because in the next induction step they will become $(s_{(M+1)1}, s_{(M+1)2}, s_{(M+1)})$.*

4.3.1 The case of two iterated l^p spaces

We will be proving the following two propositions:

Proposition 14.

$$\begin{aligned} & \left\| \left(\sum_l \left(\sum_k |BHT(f_{kl}, g_{kl})|^r \right)^{s/r} \right)^{1/s} \right\|_t \leq \\ & \leq C \left\| \left(\sum_l \left(\sum_k |f_{kl}|^{r_1} \right)^{s_1/r_1} \right)^{1/s_1} \right\|_p \cdot \left\| \left(\sum_l \left(\sum_k |g_{kl}|^{r_2} \right)^{s_2/r_2} \right)^{1/s_2} \right\|_q \end{aligned}$$

Proposition 15.

$$\begin{aligned} & \left\| \left(\sum_{k=1}^N |BHT_{\mathbb{P}(I_0)}(f_k \cdot \mathbf{1}_F, g_k \cdot \mathbf{1}_G)|^r \right)^{1/r} \cdot \mathbf{1}_{H'} \right\|_s \leq \\ & \leq \tilde{C} \left\| \left(\sum_{k=1}^N |f_k|^{r_1} \right)^{1/r_1} \right\|_p \cdot \tilde{\chi}_{I_0} \|_p \cdot \left\| \left(\sum_{k=1}^N |g_k|^{r_2} \right)^{1/r_2} \right\|_q \cdot \tilde{\chi}_{I_0} \|_q. \end{aligned}$$

where \tilde{C} has an expression similar to that in 4.22.

Proof. This is going to be a refinement of the proof of the main theorem from section 4.2. In constructing the collection of intervals $\mathcal{J}_j^{n_j}$, we note that we only need to select intervals I that are already contained in I_0 , because all the tiles in $\mathbb{P}(I_0)$ are so that $I_P \subseteq I_0$.

As before, we prove generalized restricted type estimates, and we assume the functions have the following properties:

$$\left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \leq \mathbf{1}_{E_1}, \quad \left(\sum_k |f_k|^{r_2} \right)^{1/r_2} \leq \mathbf{1}_{E_2}, \quad \left(\sum_k |h_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_{E'_3}.$$

The exceptional set is defined by $\tilde{\Omega} = \{\mathcal{M}\mathbf{1}_{E_1} > C \frac{|E_1|}{|E_3|}\} \cup \{\mathcal{M}\mathbf{1}_{E_2} > C \frac{|E_2|}{|E_3|}\}$, and we assume the tiles to be so that $1 + \frac{\text{dist}(I_P, \tilde{\Omega}^c)}{|I_P|} \sim 2^d$.

For intervals $I \in \mathcal{J}_1^{n_1}$ we have

$$\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \sim \widetilde{\text{size}}_{\mathbb{P}_{n_1}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \sim 2^{-n_1} \leq 2^d \frac{|E_1|}{|E_3|}.$$

When we consider intervals $I \in \mathcal{J}_1^{n_1} \cap \mathcal{J}_2^{n_2} \cap \mathcal{J}_3^{n_3}$, the above approximations become inequalities. We also need to point out that

$$\widetilde{\text{size}}_{\mathbb{P}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \quad \text{and} \quad \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \leq \widetilde{\text{size}}_{\mathbb{P}(I_0)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F).$$

Now we add the trilinear forms in order to obtain generalized restricted type estimates:

$$\begin{aligned} \sum |\Lambda_{BHT; \mathbb{P}(I_0)}(f_k \cdot \mathbf{1}_F, g_k \cdot \mathbf{1}_G, h_k \cdot \mathbf{1}_{H'})| &\leq \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1} \mathcal{J}_2^{n_2} \mathcal{J}_3^{n_3}} \sum_k |\Lambda_{BHT; \mathbb{P}(I_0 \cap I)}(f_k \cdot \mathbf{1}_F, g_k \cdot \mathbf{1}_G, h_k \cdot \mathbf{1}_{H'})| \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1} \mathcal{J}_2^{n_2} \mathcal{J}_3^{n_3}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{r_1}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{E_2} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{r_2}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{E'_3} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{r'}} \cdot \\ &\quad \frac{\|\mathbf{1}_{E_1} \cdot \mathbf{1}_F \cdot \tilde{\chi}_I\|_{r_1}}{|I|^{1/r_1}} \cdot \frac{\|\mathbf{1}_{E_2} \cdot \mathbf{1}_G \cdot \tilde{\chi}_I\|_{r_2}}{|I|^{1/r_2}} \cdot \frac{\|\mathbf{1}_{E'_3} \cdot \mathbf{1}_{H'} \cdot \tilde{\chi}_I\|_{r'}}{|I|^{1/r'}} |I| \lesssim \\ &\lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{E_2} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{q}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{E'_3} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{s'} - \epsilon} \cdot \\ &\quad \cdot \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1} \mathcal{J}_2^{n_2} \mathcal{J}_3^{n_3}} 2^{-\frac{n_1}{p}} 2^{-\frac{n_2}{q}} 2^{-n_3(\frac{1}{s'} + \epsilon)} |I| \end{aligned}$$

The last part adds up to something $\lesssim 2^{-\tilde{M}d} |E_1|^{\frac{1}{p}} |E_2|^{\frac{1}{q}} |E_3|^{\frac{1}{s'}}$, which is precisely what we were aiming in the beginning.

The cases when one of the r_1, r_2 or $r' = \infty$ follow in a similar manner. \square

4.3.2 Proof of Proposition 14

Proof. Once again, we use generalized restricted type interpolation; F, G, H are sets of finite measure, with $|H| = 1$. The exceptional set is defined as usual, and $H' = H \setminus \Omega$. The sequences of functions will be so that

$$\left(\sum_l \left(\sum_k |f_{kl}|^{r_1} \right)^{\frac{s_1}{r_1}} \right)^{\frac{1}{s_1}} \leq \mathbf{1}_F, \quad \left(\sum_l \left(\sum_k |g_{kl}|^{r_2} \right)^{\frac{s_2}{r_2}} \right)^{\frac{1}{s_2}} \leq \mathbf{1}_G, \quad \left(\sum_l \left(\sum_k |h_{kl}|^{r'} \right)^{\frac{s'}{r'}} \right)^{\frac{1}{s'}} \leq \mathbf{1}_{H'}.$$

The collections $\mathcal{J}_j^{n_j}$ are going to be chosen in the same way as in the proof of theorem 20, depending on the sizes and averages of the characteristic functions $\mathbf{1}_F, \mathbf{1}_G, \mathbf{1}_{H'}$. Proposition 15 yields the following:

$$\sum_k |\Lambda_{BHT; \mathbb{P}(I)}(f_{kl}, g_{kl}, h_{kl})| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{s_1}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{s_2}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{s'}}. \quad (4.23)$$

$$\cdot \left\| \left(\sum_k |f_{kl}|^{r_1} \right)^{\frac{1}{r_1}} \right\|_{s_1} \cdot \left\| \left(\sum_k |g_{kl}|^{r_2} \right)^{\frac{1}{r_2}} \right\|_{s_2} \cdot \left\| \left(\sum_k |h_{kl}|^{r'} \right)^{\frac{1}{r'}} \right\|_{s'} \cdot \tilde{\chi}_I \|_{s'} \quad (4.24)$$

When summing 4.24 over l as well, we apply Hölder for the triple s_1, s_2, s' , recovering in this way $\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{s_1}$, and the corresponding quantities for the second

and third entries. In this way we have

$$\begin{aligned}
& \left| \sum_{k,l} \Lambda_{BHT}(f_{kl}, g_{kl}, h_{kl}) \right| \lesssim \\
& \lesssim \sum_{n_1, n_2, n_3} \sum_{\mathcal{J}^{n_1, n_2, n_3}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{s_1}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{s_2}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{s'}} \cdot \\
& \quad \cdot \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{s_1}}{|I|^{1/s_1}} \cdot \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_I\|_{s_2}}{|I|^{1/s_2}} \cdot \frac{\|\mathbf{1}_{H'} \cdot \tilde{\chi}_I\|_{s'}}{|I|^{1/s'}} \cdot |I| \lesssim \\
& \lesssim \sum_{n_1, n_2, n_3} \sum_{\mathcal{J}^{n_1, n_2, n_3}} \left(\text{“size”}_{\mathbb{P}(I)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \cdot \left(\text{“size”}_{\mathbb{P}(I)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{q}} \cdot \left(\text{“size”}_{\mathbb{P}(I)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{p'} - \epsilon} \cdot \\
& \quad \cdot 2^{-\frac{n_1}{p}} 2^{-\frac{n_2}{q}} 2^{-n_3(\frac{1}{p'} + \epsilon)} \cdot |I|.
\end{aligned}$$

Remark 12. The “sizes” appearing in the line above do not represent exactly the same expression as in the initial size definition. They are max between the former sizes on I , and simpler averages on I . Nevertheless, this is the step in the proof where we can prove also the localized version of the statement in 14. Assuming all the tiles are sitting above an interval I_0 (i.e. $I_P \subseteq I_0$), we can obtain the same result with operatorial norm

$$\left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G \right)^{\frac{1+\theta_2}{2} - \frac{1}{q}} \cdot \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'} \right)^{\frac{1+\theta_3}{2} - \frac{1}{p'}}.$$

The rest of the proof is identical to the simple vector case; the quantities on *LHS* add up to $|F|^{1/p} |G|^{1/q}$, provided

$$\frac{1 + \theta_1}{2} > \frac{1}{p}, \quad \frac{1 + \theta_2}{2} > \frac{1}{q}, \quad \frac{1 + \theta_3}{2} > \frac{1}{s'}.$$

CHAPTER 5

SIMILAR RESULTS FOR PARAPRODUCTS

Recall that a discretized paraproduct is the following expression

$$\Pi(f, g)(x) = \sum_{I \in \mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1 \rangle \langle g, \phi_I^2 \rangle \phi_I^3(x),$$

where two of the families $(\phi_I^j)_{I \in \mathcal{I}}$ are *lacunary* (ϕ_I^j is a wave packet on $I \times [\frac{1}{|I|}, \frac{2}{|I|}]$), and the third one is *non-lacunary* ($\phi_I^{j_0}$ is a wave packet on $I \times [0, \frac{1}{|I|}]$).

A vector-valued extension for paraproducts reads as follows:

Theorem 22.

$$\left\| \left(\sum_k |\Pi(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q,$$

whenever $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$, and $1 < p, q \leq \infty, 0 < s < \infty$.

Remark 13. 1. Note that the range is the same as that for classical paraproducts.

So unlike the BHT extensions, we get the full possible range for vector valued extensions of paraproducts.

2. We could formulate and prove a similar result associated to families of paraproducts:

$$\left\| \left(\sum_k |\Pi_k(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q.$$

The linear form associated with this operator is

$$\Lambda_{\vec{\Pi}_r}(f, g, h) = \left\langle \left(\sum_k |\Pi_k(f_k, g_k)|^r \right)^{1/r}, h \right\rangle = \sum_k \Lambda_{\Pi_k}(f_k, g_k, h_k)$$

where $h_k = \epsilon_k(x) \cdot h(x)$, $\sum_k |\epsilon_k(x)|^{r'} = 1$ a.e. As usual, we are using interpolation of restricted weak type estimates.

While Proposition 10 is the main ingredient, we need "localized" estimates. If I_0 is some fixed dyadic interval, then we define

$$\Pi(I_0)(f, g)(x) = \sum_{\substack{I \in \mathcal{I} \\ I \subseteq I_0}} \frac{1}{|I|^{1/2}} \langle f, \phi_I^1 \rangle \langle g, \phi_I^2 \rangle \phi_I^3(x).$$

The paraproducts are far less complicated when compared with the Bilinear Hilbert Transform (they are defined over rank 0 families, versus rank 1 for *BHT*; on the other hand, the restriction of *BHT* to a tree behaves like a paraproduct). In this case we need some localization results, which will play the role of Proposition 13 and Corollary 2. However, the level of difficulty is rather comparable to that of Proposition 12, the local L^2 variant of the *BHT* localization. The reason for that is that, for paraproducts, Banach triangle estimates are easy to prove. And the same is the case with *BHT* estimates for $p, q \geq 2$.

Proposition 16 (The case $r = 1$). *Under the same assumptions as in Prop. 17,*

$$|\Lambda_{\Pi(I_0)}(f, g, \mathbf{1}_{H'})| \lesssim \widetilde{\text{size}}_{\mathcal{I}(I_0)}(\mathbf{1}_{H'}) \|f \cdot \tilde{\chi}_{I_0}\|_p \|g \cdot \tilde{\chi}_{I_0}\|_q \quad (5.1)$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$, and $1 < p, q < \infty$.

Proof. In this case $\Lambda_{\Pi(I_0)}(f, g, \mathbf{1}_{H'})$ becomes a bilinear form with respect to the first two entries. Because of the decay of $\tilde{\chi}_{I_0}$, it will be sufficient to prove the proposition in the case $\text{supp } f, g \subseteq 10I_0$. One proves restricted types estimates in this particular case and (5.1) follows by multilinear interpolation.

If $|f| \leq \mathbf{1}_F$ and $|g| \leq \mathbf{1}_G$, using Proposition 10 with $\theta_3 = 0$ and estimating $\widetilde{\text{size}}_{\mathcal{I}(I_0)}(f) \lesssim 1$, we get

$$|\Lambda_{\Pi(I_0)}(f, g, \mathbf{1}_{H'})| \lesssim \widetilde{\text{size}}_{\mathcal{I}(I_0)}(\mathbf{1}_{H'}) |F|^{\theta_1} |G|^{\theta_2}$$

where $\theta_1 + \theta_2 = 0$ and $0 < \theta_1, \theta_2 < 1$.

□

The above case corresponds to $r = 1$, and it can be used, together with Rubio de Francia's square function, to give another proof for 19.

For $r > 1$, we will have a localized paraproduct operator to which we associate a trilinear form

$$\Lambda_{\Pi(I_0)}^{\text{loc}}(f, g, h) := \Lambda_{\Pi(I_0)}(f \cdot \mathbf{1}_F, g \cdot \mathbf{1}_G, h \cdot \mathbf{1}_{H'}).$$

Proposition 17. *Let I_0 be a fixed dyadic interval and $H' \subset \mathbb{R}$ a fixed set of finite measure. Then*

$$|\Lambda_{\Pi(I_0)}^{\text{loc}}(f, g, h)| \lesssim \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_F)^{a_1} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_G)^{a_2} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_{H'})^{a_3} \cdot \|f \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h \cdot \tilde{\chi}_{I_0}\|_{r'}$$

whenever $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1$, and $1 \leq r_1, r_2, r' \leq \infty$.

Proof. The idea of the proof is very similar to that in Proposition 13. Generalized restricted type estimates will be proved by performing a stopping time. Then the result follows by interpolation.

In this case, the relation between r_1, r_2, r' and a_1, a_2, a_3 is as follows:

$$\frac{1}{r_1} = \alpha_1(1-\theta_1) + \theta_1, \quad a_1 = (1-\alpha_1)(1-\theta_1), \quad \frac{1}{r_2} = \alpha_2(1-\theta_2) + \theta_2, \quad a_2 = (1-\alpha_2)(1-\theta_2)$$

and in particular $a_1 + \frac{1}{r_1} = 1$, $a_2 + \frac{1}{r_2} = 1$. Also,

$$\frac{1}{r'} = \theta_3 - \alpha_1(1-\theta_1) - \alpha_2(1-\theta_2), \quad a_3 = (1-\alpha_3)(1-\theta_3).$$

□

5.0.3 Proof of Theorem 22

Using vector-valued interpolation theorems, we fix F, G and H sets of finite measure; we assume $H = 1$. Let $f = (f_k)_k, g = (g_k)_k$, with $(\sum_k |f_k|^{r_1})^{1/r_1} \leq \mathbf{1}_F$, $(\sum_k |g_k|^{r_2})^{1/r_2} \leq \mathbf{1}_G$.

The exceptional set will be

$$\tilde{\Omega} := \{x : \mathcal{M}(\mathbf{1}_F)(x) > C|F|\} \cup \{x : \mathcal{M}(\mathbf{1}_G)(x) > C|G|\}$$

and $H' = H \setminus \tilde{\Omega}$. We have a function $h \leq \mathbf{1}_{H'}$ and $h_k(x) = h(x) \cdot \epsilon_k(x)$, with $\|h_k(x)\|_{l'_k} = 1$ a.e.

For every $d \geq 0$

$$\mathcal{J}^d := \{I \in \mathcal{J} : 1 + \frac{\text{dist}(I, \Omega^c)}{|I|} \sim 2^d\}$$

When estimating paraproducts associated to the collection \mathcal{J}_k^d we get an extra 2^{-10d} decay and thus the d -dependency of the paraproducts can be assumed to be implicit. As before, for each of the sets F, G and H' we define collections of disjoint maximal intervals $\mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}$ and $\mathcal{J}_3^{n_3}$ respectively. For example, if $I \in \mathcal{J}_1^{n_1}$, then

$$2^{-n_1-1} \leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_F \cdot \tilde{\chi}_I dx \leq 2^{-n_1} \lesssim |F|.$$

Returning to the operator $\vec{\Pi}_r$ and its multilinear form, , we have

$$\begin{aligned} |\sum_k \Lambda_{\Pi_k}(f_k, g_k, h_k)| &\leq \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}, \mathcal{J}_3^{n_3}} \sum_k |\Lambda_{\Pi_k(I)}(f_k, g_k, h_k)| \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}, \mathcal{J}_3^{n_3}} \sum_{k=1}^n \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_F)^{b_1} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_G)^{b_2} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_{H'})^{b_3} \|f_k \cdot \tilde{\chi}_I\|_{r_1} \|g_k \cdot \tilde{\chi}_I\|_{r_2} \|h_k \cdot \tilde{\chi}_I\|_{r'} \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I \in \mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}, \mathcal{J}_3^{n_3}} \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_F)^{b_1} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_G)^{b_2} \cdot \widetilde{\text{size}}_{\mathcal{J}(I_0)}(\mathbf{1}_{H'})^{b_3} \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{r_1}}{|I|^{1/r_1}} \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_I\|_{r_2}}{|I|^{1/r_2}} \frac{\|\mathbf{1}_{H'} \cdot \tilde{\chi}_I\|_{r'}}{|I|^{1/r'}} |I| \end{aligned}$$

Here we let $0 \leq b_j \leq a_j$, because the sizes are anyways subunitary. Whenever

$0 \leq \gamma_j \leq 1$ are so that $\gamma_1 + \gamma_2 + \gamma_3 = 1$,

$$\sum_{I \in \mathcal{J}^{n_1, n_2, n_3}} |I| \lesssim (2^{n_1} |F|)^{\gamma_1} (2^{n_2} |G|)^{\gamma_2} (2^{n_3} |H|)^{\gamma_3}.$$

Adding all the pieces together we have

$$|\sum_k \Lambda_{\Pi_k}(f_k, g_k, h_k)| \lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(b_1 + \frac{1}{p} - \gamma_1)} 2^{-n_2(b_2 + \frac{1}{q} - \gamma_2)} 2^{-n_3(b_3 + \frac{1}{r} - \gamma_3)} |F|^{\gamma_1} |G|^{\gamma_2} \lesssim |F|^{\frac{1}{p}} |G|^{\frac{1}{q}}.$$

Of course, the last inequality is true provided we can choose $\gamma_1, \gamma_2, \gamma_3$ so that the series converges. Choosing the θ_j s and α_j s carefully, one can prove that the restricted weak type estimates hold arbitrary close to the points

$$(0, 0, 1), (1, 0, 0), (0, 1, 0), \text{ and } (1, 1, -1).$$

Then the result follows by interpolation. □

5.0.4 Generalized Paraproducts

A more generalized version of a paraproduct is

$$\tilde{\Pi}^\alpha(f, g)(x) = \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \varphi_I^{3, \alpha}(x),$$

where $\varphi_I^{3, \alpha}$ is L^2 -normalized and has slow decay:

$$||I|^{1/2} \varphi_I^{3, \alpha}(x)| \lesssim \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{(1+\alpha)}}. \quad (5.2)$$

The Banach Triangle case

It is much more easy to prove that $\tilde{\Pi}^\alpha : L^p \times L^q \rightarrow L^s$ when $1 < p, q < \infty$, and $1 \leq s < \infty$. The proof only uses the boundedness of the discrete square

functions and maximal operator (even though it is associated with a family of functions that decay relatively slow as $|x| \rightarrow \infty$). We illustrate this ideas:

$$\begin{aligned}
|\int_{\mathbb{R}} \tilde{\Pi}^\alpha(f, g)(x) h(x) dx| &= |\sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \varphi_I^{3, \alpha} \rangle| = \\
&= |\int \sum_{I \in \mathcal{J}} \frac{\langle f, \psi_I \rangle}{|I|^{1/2}} \cdot \mathbf{1}_I(x) \frac{\langle g, \psi_I \rangle}{|I|^{1/2}} \cdot \mathbf{1}_I(x) \frac{\langle h, \varphi_I^{3, \alpha} \mathbf{1}_I(x) \rangle}{|I|^{1/2}} dx| \lesssim \\
&\lesssim |\int \mathcal{S}(f)(x) \cdot \mathcal{S}(g)(x) \cdot \mathcal{M}^\alpha(h)(x) dx| \lesssim \\
&\lesssim \|f\|_p \cdot \|g\|_q \cdot \|h\|_{s'}
\end{aligned}$$

The case when $p = \infty$ or $s < 1$ is slightly more difficult, but can be proved in a way similar to the discretized paraproduct case, using the *sizes* and *energies* quantities. However, in this case, we have the constraint

$$\frac{1}{1 + \alpha} < s < \infty, \quad \text{or equivalently, } -\alpha < \frac{1}{s'}.$$

It's not only that the techniques break down, but one can actually prove that this is a necessary condition (see [11]).

An approach one can follow is to break down each $\varphi_I^{3, \alpha}$ into pieces that are compactly supported:

$$\varphi_I^{3, \alpha} = \sum_{l \geq 0} \frac{1}{2^{l(\alpha-)}} \varphi_I^{3, \alpha, l}(x), \quad (5.3)$$

where each $\varphi_I^{3, \alpha, l}$ is supported inside $2^{l+2}I$ and

$$|I|^{1/2} \varphi_I^{3, \alpha, l}(x) \lesssim \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{(1+)}}.$$

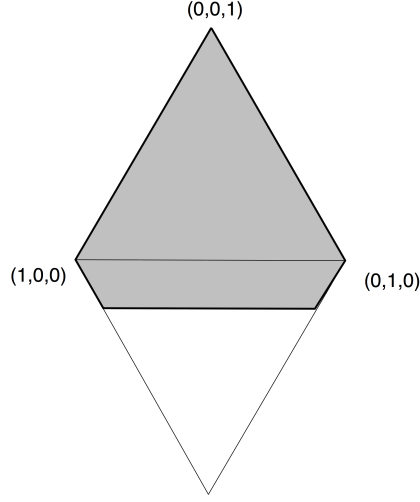


Figure 5.1: Range of $\tilde{\Pi}^\alpha$: restriction $\frac{1}{s'} > \alpha$.

In some sense, the compact support of the $\varphi_I^{3,\alpha,l}$ compensate for the slow decay of the $\varphi_I^{3,\alpha}$. Also, one can use L^p -adapted energies, as in section 3.1, where we had L^2 -adapted energies.

Proof. However, here we present a slightly different proof. Let E_1, E_2 , and E_3 be sets so that $|E_3| = 1$. The exceptional set is given by

$$\Omega = \{x : \mathcal{M}\mathbf{1}_{E_1} > C \frac{|E_1|}{|E_3|}\} \cup \{x : \mathcal{M}\mathbf{1}_{E_2} > C \frac{|E_2|}{|E_3|}\}.$$

Then $E'_3 = E_3 \setminus \Omega$. The functions f, g, h are so that $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E_2}, |h| \leq \mathbf{1}_{E'_3}$. Assume the intervals $I \in \mathcal{I}$ are so that $\frac{\text{dist}(I, \Omega^c)}{|I|} \sim 2^d$. Using Proposition 10, we have

$$\begin{aligned} \left| \sum_{I \in \mathcal{I}} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \psi_I^{3,\alpha} \rangle \right| &\lesssim (\text{size } \mathbf{1}_{E_1})^{1-\theta_1} \cdot (\text{size } \mathbf{1}_{E_2})^{1-\theta_2} \cdot (\text{size } \mathbf{1}_{E'_3})^{1-\theta_3} \\ &\quad \cdot (\text{energy } \mathbf{1}_{E_1})^{\theta_1} \cdot (\text{energy } \mathbf{1}_{E_2})^{\theta_2} \cdot (\text{energy } \mathbf{1}_{E'_3})^{\theta_3}. \end{aligned}$$

Observe the following:

- size $\mathbf{1}_{E_1} \lesssim \min\{1, 2^d |E_1|\}$ and size $\mathbf{1}_{E_2} \lesssim \min\{1, 2^d |E_2|\}$.

•

$$\text{size } \mathbf{1}_{E'_3} \lesssim \sup_I \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E'_3} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\alpha}} dx \lesssim 2^{-d\alpha}.$$

This implies that

$$\begin{aligned} \left| \sum_{I \in \mathcal{J}} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \psi_I^{3,\alpha} \rangle \right| &\lesssim \left(2^d |E_1|\right)^{\nu_1} \cdot \left(2^d |E_1|\right)^{\nu_1} \cdot 2^{-d\alpha(1-\theta_3)} |E_1|^{\theta_1} |E_2|^{\theta_2} = \\ &= 2^{-d(\alpha(1-\theta_3)-\nu_1-\nu_2)} |E_1|^{\nu_1+\theta_1} |E_2|^{\nu_2+\theta_2}. \end{aligned}$$

We will want $\frac{1}{p} = \nu_1 + \theta_1$, $\frac{1}{q} = \nu_2 + \theta_2$. Also,

$$\alpha(1 - \theta_3) > \nu_1 + \nu_2 = \frac{1}{p} + \frac{1}{q} - (1 - \theta_3) \Leftrightarrow (1 + \alpha)(1 - \theta_3) > \frac{1}{s}.$$

In the end, this last condition implies that $\frac{1}{1 + \alpha} < s < \infty$. □

Localization Lemma

In this section we use two types of bump functions; as before, $\tilde{\chi}_{I_0}$ is adapted to the interval I_0 , while ϕ_{I_0} is only *weakly adapted* in the sense that

$$\phi_{I_0}(x) = \frac{1}{\left(1 + \frac{\text{dist}(x, I_0)}{|I_0|}\right)^{\#}}.$$

The variable $\#$ will depend on the context of the theorem, and will be explicitly defined later on.

Define

$$\tilde{\Pi}_{\text{loc}}^{\alpha}(f, g)(x) = \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{1}{|I|^{1/2}} \langle f \cdot \mathbf{1}_F, \psi_I \rangle \langle g \cdot \mathbf{1}_G, \psi_I \rangle \varphi_I^{3,\alpha}(x) \cdot \mathbf{1}_{H'}(x).$$

Lemma 7. Let $1 < r_1, r_2, r < \infty$ be so that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$. Then one can find $0 \leq a_1, a_2, a_3 < 1$ so that

$$\left| \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{1}{|I|^{1/2}} \langle f \cdot \mathbf{1}_F, \psi_I \rangle \langle g \cdot \mathbf{1}_G, \psi_I \rangle \langle h \cdot \mathbf{1}_{H'}, \varphi_I^{3,\alpha} \rangle \right| \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{a_3}.$$

$$\|f \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h \cdot \phi_{I_0}\|_{r'}.$$

Proof. We use restricted weak type interpolation (we don't need generalized restricted type interpolation because we are inside the Banach triangle). Let E_1, E_2 and E_3 be sets of finite measure, supported inside $5I_0$; we will find a major subset $E'_3 \subset E_3$ so that

$$|\Lambda_{\tilde{\Gamma}_{\text{loc}}}^\alpha(\mathbf{1}_{E_1}, \mathbf{1}_{E_2}, \mathbf{1}_{H'})| \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{a_3} |E_1|^{1/r_1} |E_2|^{1/r_2} |E_3|^{1/r'}.$$

Moreover, when E_3 is supported on $2^{m+1}I_0 \setminus 2^mI_0$, there is an extra $2^{-m\#}$ term appearing on the *RHS*, which justifies the term $\|h \cdot \phi_{I_0}\|_{r'}$. The value of $\#$ and the role it plays becomes clear as we proceed with the proof. So from now on we assume E_3 is supported on the shell $2^{m+1}I_0 \setminus 2^mI_0$.

Similarly one obtains the terms $\|f \cdot \tilde{\chi}_{I_0}\|_p$ and $\|g \cdot \tilde{\chi}_{I_0}\|_q$. The fast decay of the Schwartz functions enables us to take $a_1 = \frac{1}{r'_1} - \epsilon$ and $a_2 = \frac{1}{r'_2} - \epsilon$.

The construction of the exceptional set is rather standard; now we let $|E_3|$ be any number.

$$\Omega = \left\{ x : \mathcal{M}\mathbf{1}_{E_1} > C \frac{|E_1|}{|E_3|} \right\} \cup \left\{ x : \mathcal{M}\mathbf{1}_{E_2} > C \frac{|E_2|}{|E_3|} \right\}.$$

Also, we assume that all the intervals $I \subseteq I_0$ considered above have the prop-

erty that $\frac{\text{dist}(I, \Omega^c)}{|I|} \sim 2^d$. This step is not necessary in the case $1 < r_1, r_2, r' < \infty$.

$$\begin{aligned} \left| \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \psi_I^{3,\alpha} \rangle \right| &\lesssim (\text{size } \mathbf{1}_F \cdot \mathbf{1}_{E_1})^{1-\theta_1} \cdot (\text{size } \mathbf{1}_G \cdot \mathbf{1}_{E_2})^{1-\theta_2} \cdot (\text{size } \mathbf{1}_{H'} \cdot \mathbf{1}_{E_3'})^{1-\theta_3} \\ &\quad \cdot (\text{energy } \mathbf{1}_F \cdot \mathbf{1}_{E_1})^{\theta_1} \cdot (\text{energy } \mathbf{1}_G \cdot \mathbf{1}_{E_2})^{\theta_2} \cdot (\text{energy } \mathbf{1}_{H'} \cdot \mathbf{1}_{E_3'})^{\theta_3} \end{aligned}$$

The RHS can be further estimated by

$$\begin{aligned} &(\text{size } \mathbf{1}_F)^{a_1} \cdot (\text{size } \mathbf{1}_G)^{a_2} \cdot (\text{size } \mathbf{1}_{H'})^{a_3} (\text{size } \mathbf{1}_{E_1})^{b_1} \cdot (\text{size } \mathbf{1}_{E_2})^{b_2} \cdot (\text{size } \mathbf{1}_{E_3'})^{b_3} \\ &\quad \cdot |E_1|^{\theta_1} \cdot |E_2|^{\theta_2} \cdot |E_3|^{\theta_3}. \end{aligned}$$

Here $a_i + b_i = 1 - \theta_i$; the *size* corresponding to $\mathbf{1}_{E_3'}$ is majorized by

$$\text{size } \mathbf{1}_{E_3'} \lesssim \sup_{I \subseteq I_0} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_3'} \cdot \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\alpha}} dx \lesssim \min\{2^{-d\alpha}, 2^{-m\alpha}\}.$$

$$\begin{aligned} \left| \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \psi_I^{3,\alpha} \rangle \right| &\lesssim (\widetilde{\text{size}} \mathbf{1}_F)^{a_1} \cdot (\widetilde{\text{size}} \mathbf{1}_G)^{a_2} \cdot (\widetilde{\text{size}} \mathbf{1}_{H'})^{a_3} \\ &\quad \cdot \left(2^d \frac{|E_1|}{|E_3|}\right)^{b_1} \cdot \left(2^d \frac{|E_2|}{|E_3|}\right)^{b_2} \cdot 2^{-d\alpha b_{31}} \cdot 2^{-m\alpha b_{32}} \cdot |E_1|^{\theta_1} \cdot |E_2|^{\theta_2} \cdot |E_3|^{\theta_3} \lesssim \\ &\lesssim (\widetilde{\text{size}} \mathbf{1}_F)^{a_1} \cdot (\widetilde{\text{size}} \mathbf{1}_G)^{a_2} \cdot (\widetilde{\text{size}} \mathbf{1}_{H'})^{a_3} \cdot 2^{-d(\alpha b_{31} - b_1 - b_2)} \cdot 2^{-m\alpha b_{32}} \\ &\quad \cdot |E_1|^{b_1 + \theta_1} |E_2|^{b_2 + \theta_2} |E_3|^{\theta_3 - b_1 - b_2} \end{aligned}$$

We are mainly interested in two cases, and we will focus our analysis on those:

$r' > 0$ and $r' = 0$.

1. if $0 < r_1, r_2, r' < \infty$, we can choose $\theta_1 = \frac{1}{r_1}$, $\theta_2 = \frac{1}{r_2}$, $b_1 = b_2 = b_{31} = 0$.

Then $a_3 + b_{32} = 1$.

2. The case $r' = 0$, and consequently $\frac{1}{r_1} + \frac{1}{r_2} = 1$ will be obtained by interpolation between the points

$$\left(\frac{1}{r_1} - 3\eta, \frac{1}{r_2} + 2\eta, \eta\right), \left(\frac{1}{r_1} + 2\eta, \frac{1}{r_2} - 3\eta, \eta\right), \left(\frac{1}{r_1} - \eta, \frac{1}{r_2} + 2\eta, -\eta\right), \left(\frac{1}{r_1} + 2\eta, \frac{1}{r_2} - \eta, -\eta\right).$$

The first two are strictly contained in the Banach triangle for η sufficiently small, so we need only deal with the last two. The two points are quite symmetric, so it is enough to deal with the last one $\left(\frac{1}{r_1} + 2\eta, \frac{1}{r_2} - \eta, -\eta\right)$. In this case, we can pick

$$\theta_1 = \frac{1}{r_1} - \eta, \quad \theta_2 = \frac{1}{r_2} - \eta, \quad \theta_3 = 2\eta, \quad b_1 = 3\eta, \quad b_2 = 0, \quad b_{31} = \frac{4\eta}{\alpha}.$$

Note that $b_{32} + a_3 = 1 - \frac{4\eta}{\alpha}$.

This proves the lemma in the two cases we care about, and moreover we note that $a_i = \frac{1}{r_i} - \epsilon$, $a_3 + b_{32} \sim 1$. The ϕ_{l_0} is associated to $\# = \alpha b_{32}$.

□

Vector-valued estimates for $\tilde{\Pi}^\alpha$: the case $r' > 0$

Theorem 23. *If $1 < r < \infty$, $1 < r_1, r_2 \leq \infty$, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$, and $\alpha r' > 1$, then there exists a non-empty range $\mathcal{D}_r(\alpha)$ for which*

$$\left\| \left(\sum_k |\tilde{\Pi}^\alpha|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \cdot \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q.$$

We then say that $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'}\right) \in \mathcal{D}_r(\alpha)$. The range can be described by the conditions

$$-\lambda \left(a_3 + \frac{1}{r'} \right) < \frac{1}{s'} < 1 - \frac{1}{\alpha r'} + \frac{1}{r'},$$

where a_3 is so that $0 < a_3 < 1 - \frac{1}{\alpha r'}$, and λ implicitly depends on a_3 as well. For example, we can take a_3 arbitrarily close to $1 - \frac{1}{r'\alpha}$; in that case λ will be very close to 0. In general, the picture that we have for $\mathcal{D}_r(\alpha)$ is

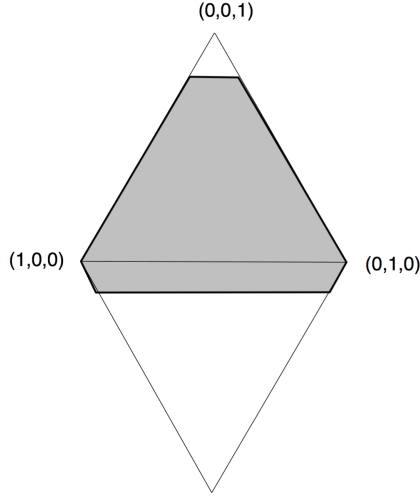


Figure 5.2: Range of $\tilde{\Pi}^\alpha$: restriction $-\lambda \left(a_3 + \frac{1}{r'} \right) < \frac{1}{s'} < 1 - \frac{1}{\alpha r'} + \frac{1}{r'}$.

Proof. Let $\left(\sum |f_k|^{r_1} \right)^{1/r_1} \leq \mathbf{1}_F$, $\left(\sum |g_k|^{r_2} \right)^{1/r_2} \leq \mathbf{1}_G$ and $\left(\sum |h_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_{H'}$, with $|H| = 1$.

Here the exceptional set is

$$\Omega = \{x : \mathcal{M}\mathbf{1}_F(x) > C|F|\} \cup \{x : \mathcal{M}\mathbf{1}_G(x) > C|G|\}.$$

For every function $\mathbf{1}_F, \mathbf{1}_G$ and $\mathbf{1}_{H'}$, we will have families of partitions $\{\mathcal{J}_{n_i}\}_{n_i}$. The constructions of these will be explained later on. We also denote $\mathcal{J}^{n_1, n_2, n_3} = \mathcal{J}_{n_1} \cap \mathcal{J}_{n_2} \cap \mathcal{J}_{n_3}$. The trilinear form associated with the vector-valued operator breaks down in smaller pieces, for which we have better control:

$$\begin{aligned} \left| \sum_k \Lambda_{\tilde{\Pi}^\alpha}(f_k, g_k, h_k) \right| &\leq \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} \left| \sum_k \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f_k, \psi_I \rangle \langle g_k, \psi_I \rangle \langle h_k, \varphi_I^{3, \alpha} \rangle \right| \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} \sum_k \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{a_3} \cdot \\ &\quad \cdot \|f_k \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g_k \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h_k \cdot \phi_{I_0}\|_{r'}. \end{aligned}$$

Applying Hölder, we have

$$\begin{aligned} |\sum_k \Lambda_{\tilde{\Pi}^\alpha}(f_k, g_k, h_k)| &\lesssim \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} (\widetilde{\text{size}}_{I_0} \mathbf{1}_F)^{a_1} \cdot (\widetilde{\text{size}}_{I_0} \mathbf{1}_G)^{a_2} \cdot (\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'})^{a_3} \\ &\cdot \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\|_{r_1}}{|I_0|^{r_1}} \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\|_{r_2}}{|I_0|^{r_2}} \frac{\|\mathbf{1}_{H'} \cdot \phi_{I_0}\|_{r'}}{|I_0|^{r'}} \cdot |I_0|. \end{aligned}$$

Here we need to discuss in more detail the partitions \mathcal{J}_{n_i} and compare the *sizes* and averages. For $\mathbf{1}_F$ (and almost identically for $\mathbf{1}_G$), the collection \mathcal{J}_{n_1} contains maximal dyadic intervals

$$\mathcal{J}_{n_1} : \quad I \text{ such that } 2^{-n_1-1} \leq \int_{\mathbb{R}} \mathbf{1}_F \cdot \tilde{\chi}_I(x) dx \leq 2^{-n_1} \lesssim 2^d |F|.$$

One notices that $\sum_{I \in \mathcal{J}_{n_1}} |I| \lesssim 2^{n_1} |F|$.

On the other hand, for $\mathbf{1}_{H'}$ we need to compare

$$\begin{aligned} \widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} &:= \sup_{I \subseteq I_0} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\alpha}} dx \quad \text{and} \\ \frac{\|\mathbf{1}_{H'} \cdot \phi_{I_0}\|_{r'}}{|I_0|} &= \frac{1}{|I_0|} \int_{\mathbb{R}} \mathbf{1}_{H'} \cdot \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{\#r'}} dx. \end{aligned}$$

If we pick b_{32} so that $\#r' > 1$, the expression above is integrable, and maximal operators associated with $\frac{1}{(1 + |x|)^{\#r'}}$ exist. For this to be possible, one would need $\alpha \geq ab_{32} > \frac{1}{r'}$. Let $1 + \lambda = \min\{1 + \alpha, \#r'\} > 1$. The stopping time will be done with respect to

$$2^{-n_3-1} \leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \cdot \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\lambda}} dx \leq 2^{-n_3} \lesssim 2^{-d\lambda}.$$

In this case we have $\sum_{I \in \mathcal{J}_{n_3}} |I| \leq 2^{n_3}$.

$$\begin{aligned} |\sum_k \Lambda_{\tilde{\Pi}^\alpha}(f_k, g_k, h_k)| &\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(a_1 + \frac{1}{r_1} - \theta_1)} 2^{-n_2(a_2 + \frac{1}{r_2} - \theta_2)} 2^{-n_3(a_3 + \frac{1}{r'} - \theta_3)} |F|^{\theta_1} |G|^{\theta_2} \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(v_1 - \theta_1)} 2^{-n_2(v_2 - \theta_2)} 2^{-n_3(a_3 + \frac{1}{r'} - \theta_3)} |F|^{\theta_1} |G|^{\theta_2}, \end{aligned}$$

where, because the *sizes* are $\lesssim 1$, we can take

$$\nu_1 \leq a_1 + \frac{1}{r_1}, \quad \nu_2 \leq a_2 + \frac{1}{r_2}.$$

We need them to satisfy the condition $\nu_1 + \nu_2 + a_3 + \frac{1}{r'} > 1$. If this is satisfied, we have

$$\begin{aligned} |\sum_k \Lambda_{\tilde{\Gamma}^\alpha}(f_k, g_k, h_k)| &\lesssim (2^d |F|)^{\nu_1 - \theta_1} \cdot (2^d |G|)^{\nu_2 - \theta_2} 2^{-d\lambda(a_3 + \frac{1}{r'} - \theta_3)} |F|^{\theta_1} |G|^{\theta_2} \lesssim \\ &\lesssim 2^{-d[\lambda(a_3 + \frac{1}{r'} - \theta_3) - (\nu_1 - \theta_1) - (\nu_2 - \theta_2)]} |F|^{\nu_1} |G|^{\nu_2}. \end{aligned}$$

In this case, $\nu_1 = \frac{1}{p}$, $\nu_2 = \frac{1}{q}$. The condition

$$\lambda(a_3 + \frac{1}{r'} - \theta_3) - (\nu_1 - \theta_1) - (\nu_2 - \theta_2) > 0$$

needs to hold. It becomes equivalent to

$$\lambda(a_3 + \frac{1}{r'}) - \lambda\theta_3 > \frac{1}{s} - 1 + \theta_3 \Leftrightarrow \frac{1}{s} < \lambda\left(a_3 + \frac{1}{r'}\right) + 1 - (\lambda + 1)\theta_3 < 1 + \lambda\left(a_3 + \frac{1}{r'}\right).$$

The two conditions together imply

$$-\lambda\left(1 - \frac{\#}{\alpha} + \frac{1}{r'}\right) = -\lambda\left(a_3 + \frac{1}{r'}\right) < \frac{1}{s'} < a_3 + \frac{1}{r'} = 1 - \frac{\#}{\alpha} + \frac{1}{r'}.$$

Remark 14. The expression on the right is maximal when $\#$ is minimal; so take $\# = \frac{1}{r'} + \eta$, the condition reads as

$$\frac{1}{s'} < 1 + \frac{1}{r'}\left(1 - \frac{1}{\alpha}\right) - \frac{\eta}{\alpha} < 1 + \frac{1}{r'}\left(1 - \frac{1}{\alpha}\right).$$

How close $\frac{1}{s'}$ can be to 1 depends on r' and α .

□

The case $r' = 0$

Lemma 8. *Let $1 < r_1, r_2 < \infty$ be so that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then one can find $0 \leq a_1, a_2, a_3 < 1$ arbitrarily close to $\frac{1}{r'_1}, \frac{1}{r'_2}$ and $1 - \epsilon$ respectively, for which*

$$\|\tilde{\Pi}_{loc}^\alpha(f, g)\|_1 \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F\right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G\right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'}\right)^{a_3} \|f \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g \cdot \tilde{\chi}_{I_0}\|_{r_2}.$$

Proof. The idea of the proof is very similar to the case $r' > 0$. We have $|h| \leq \mathbf{1}_{E'_3}$ and with the exactly same construction as before, we have

$$\begin{aligned} & \left| \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f, \psi_I \rangle \langle g, \psi_I \rangle \langle h, \varphi_I^\alpha \rangle \right| \lesssim \left(\widetilde{\text{size}} \mathbf{1}_F\right)^{a_1} \cdot \left(\widetilde{\text{size}} \mathbf{1}_G\right)^{a_2} \cdot \left(\widetilde{\text{size}} \mathbf{1}_{H'}\right)^{a_3} \cdot 2^{-d(\alpha b_3 - b_1 - b_2)} \\ & \cdot |E_1|^{b_1 + \theta_1} |E_2|^{b_2 + \theta_2} |E_3|^{\theta_3 - b_1 - b_2}. \end{aligned}$$

The aim is to find some set of points $\{(b_1 + \theta_1, b_2 + \theta_2, \theta_3 - b_1 - b_2)\}_{b_1, b_2, \theta_1, \theta_2}$ contained in an arbitrarily small neighborhood of $(\frac{1}{r_1}, \frac{1}{r_2}, 0)$, so that the latter lies in the convex hull of at least three of these points. We pick four points:

$$\left(\frac{1}{r_1} - \eta, \frac{1}{r_2} - \eta, 2\eta\right), \quad \left(\frac{1}{r_1} - \eta, \frac{1}{r_2} - \eta, 2\eta\right), \quad \left(\frac{1}{r_1} + 2\eta, \frac{1}{r_2} - \eta, -\eta\right), \quad \left(\frac{1}{r_1} - \eta, \frac{1}{r_2} + 2\eta, -\eta\right).$$

The first are inside the Banach triangle, so we can just take $b_1 = b_2 = 0$. Then the condition $\alpha b_3 > b_1 + b_2$ trivially holds. For the third one, we take $b_1 = b_2 = \eta$, $\theta_1 = \frac{1}{r_1} + \eta$, $\theta_2 = \frac{1}{r_2} - 2\eta$, $\theta_3 = \eta$. Then if we choose $b_3 = \frac{3\eta}{\alpha}$, the series in the d variable converges.

In all of these cases we have $a_1 = \frac{1}{r'_1} - \epsilon$, $a_2 = \frac{1}{r'_2} - \epsilon$, $a_3 = 1 - \epsilon$. \square

Now we are using the localization lemma for proving the following result:

Theorem 24. *Let $1 < r_1, r_2 < \infty$ be so that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\left\| \sum_k |\tilde{\Pi}^\alpha(f_k, g_k)| \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \cdot \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q$$

whenever $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'}\right) \in \text{Range}(\tilde{\Pi}^\alpha)$, which we denote $\mathcal{D}_1(\alpha)$.

Proof. In this case, $\sup_k |h_k(x)| \leq \mathbf{1}_{H'}$, where the exceptional set is defined as before.

$$\begin{aligned} |\sum_k \Lambda_{\tilde{\Pi}^\alpha}(f_k, g_k, h_k)| &\leq \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} \sum_k \left| \int_{\mathbb{R}} \tilde{\Pi}_{\text{loc}}^\alpha(f_k, g_k)(x) h_k(x) dx \right| \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} \sum_k \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{a_3} \|f_k \cdot \tilde{\chi}_{I_0}\|_{r_1} \|g_k \cdot \tilde{\chi}_{I_0}\|_{r_2} \|h_k\|_\infty \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{J}^{n_1, n_2, n_3}} \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{a_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{a_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{a_3} \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\|_{r_1}}{|I_0|^{1/r_1}} \cdot \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_{I_0}\|_{r_2}}{|I_0|^{1/r_2}} \cdot |I_0|. \end{aligned}$$

The stopping time for \mathcal{J}^{n_3} in this case is so that

$$2^{-n_3-1} \leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{(1+\alpha)}} dx \leq 2^{-n_3} \lesssim 2^{-d\alpha}.$$

The initial vector valued estimate becomes

$$\begin{aligned} |\sum_k \Lambda_{\tilde{\Pi}^\alpha}(f_k, g_k, h_k)| &\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(a_1 + \frac{1}{r_1} - \theta_1)} 2^{-n_2(a_2 + \frac{1}{r_2} - \theta_2)} 2^{-n_3(a_3 - \theta_3)} |F|^{\theta_1} |G|^{\theta_2} \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(\nu_1 - \theta_1)} 2^{-n_2(\nu_2 - \theta_2)} 2^{-n_3(a_3 - \theta_3)} |F|^{\theta_1} |G|^{\theta_2} \lesssim \\ &\lesssim \left(2^d |F|\right)^{\nu_1 - \theta_1} \left(2^d |G|\right)^{\nu_2 - \theta_2} \left(2^{-d\alpha}\right)^{a_3 - \theta_3} |F|^{\theta_1} |G|^{\theta_2} \lesssim \\ &\lesssim 2^{-d[\alpha(a_3 - \theta_3) - (\nu_1 - \theta_1) - (\nu_2 - \theta_2)]} |F|^{\nu_1} |G|^{\nu_2}. \end{aligned}$$

The convergence is conditioned by

$$\nu_1 + \nu_2 + a_3 > 1, \quad \alpha(a_3 - \theta_3) > \frac{1}{s} - \theta_1 - \theta_2.$$

The first condition holds because we can take a_3 arbitrarily close to 1; the second one becomes equivalent to $\frac{1}{s} < 1 + \alpha a_3 \sim 1 + \alpha$, which is the same restriction that appears in the scalar case for $\tilde{\Pi}^\alpha$.

So $\mathcal{D}_1(\alpha) = \{(\beta_1, \beta_2, \beta_3) : \beta_1 + \beta_2 + \beta_3 = 1, 0 \leq \beta_1, \beta_2 < 1 \text{ and } -\alpha < \beta_3 < 1\}$. \square

The case of two iterated vector spaces

In this section we aim to prove vector valued estimates of the type

$$\tilde{\Pi}^\alpha : L^p(I^{s_1}(I^{r_1})) \times L^q(I^{s_2}(I^{r_2})) \rightarrow L^t(I^s(I^r)). \quad (5.4)$$

However, we don't have estimates for any r and s .

$$\mathcal{D}_{r,s}(\alpha) := \left\{ \left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'} \right) : \text{for which 5.4 holds} \right\}.$$

We also denote $\mathcal{I}_2(\alpha)$ the set of ordered pairs (r, s) for which $\mathcal{D}_{r,s}(\alpha) \neq \emptyset$. For this purpose, we need a localized version of the vector-valued $\tilde{\Pi}^\alpha$ result.

Lemma 9. *Let I_0 be some dyadic interval, and F, G, H' are sets of finite measure. Then*

$$\begin{aligned} & \left| \sum_k \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f_k \cdot \mathbf{1}_F, \psi_I \rangle \langle g_k \cdot \mathbf{1}_G, \psi_I \rangle \langle h_k \cdot \mathbf{1}_{H'}, \varphi_I^{3,\alpha} \rangle \right| \lesssim \\ & \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{\alpha_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{\alpha_2} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'} \right)^{\alpha_3} \cdot \\ & \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \cdot \tilde{\chi}_{I_0} \right\|_{s_1} \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \cdot \tilde{\chi}_{I_0} \right\|_{s_2} \left\| \left(\sum_k |h_k|^{r'} \right)^{1/r'} \cdot \phi_{I_0} \right\|_{s'} \end{aligned}$$

for some $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$. In this case $\phi_{I_0}(x) = \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{\mathfrak{q}}}$, where the value of \mathfrak{q} will be established during the proof.

Proof. For the right hand side, to explain the averaging of the h_k , we are going to assume that $H' \subseteq 2^{m+1}I_0 \setminus 2^m I_0$, while $\left(\sum_k |h_k| \right)^{1/r'} \leq \mathbf{1}_{H'}$.

We need to go back to the proof of the initial statement; the notation, exceptional sets, stopping times and assumptions are identical, but now we have

$$\begin{aligned} & \left| \sum_k \Lambda_{\tilde{\Pi}_{\text{loc}}^\alpha}^\alpha(f_k, g_k, h_k) \right| \lesssim \sum_{n_1, n_2, n_3} \sum_{\substack{I \in \mathcal{I}_{n_1, n_2, n_3} \\ I \subseteq I_0}} \left(\widetilde{\text{size}}_I \mathbf{1}_F \right)^{\alpha_1} \cdot \left(\widetilde{\text{size}}_I \mathbf{1}_G \right)^{\alpha_2} \cdot \left(\widetilde{\text{size}}_I \mathbf{1}_{H'} \right)^{\alpha_3} \cdot \\ & \cdot \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{r_1}}{|I|^{1/r_1}} \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_I\|_{r_2}}{|I|^{1/r_2}} \frac{\|\mathbf{1}_{H'} \cdot \phi_I\|_{r'}}{|I|^{1/r'}} \cdot |I|. \end{aligned}$$

Again , we need to compare two quantities: $\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{\#r'}} dx$ and $\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}$. Instead, we let $1 + \lambda = \min\{1 + \alpha, \#r'\}$ and work with a similar size:

$$\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}} = \sup_{I' \subseteq I} \frac{1}{|I'|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I')}{|I'|}\right)^{1+\lambda}} dx.$$

$$\left| \sum_k \Lambda_{\tilde{\Gamma}_{\text{loc}}^\alpha}(f_k, g_k, h_k) \right| \lesssim \sum_{n_1, n_2, n_3} \sum_{\substack{I \in \mathcal{I}^{n_1, n_2, n_3} \\ I \subseteq I_0}} \left(\widetilde{\text{size}_I \mathbf{1}_F} \right)^{a_1 + \frac{1}{r_1}} \cdot \left(\widetilde{\text{size}_I \mathbf{1}_G} \right)^{a_2 + \frac{1}{r_2}} \cdot \left(\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}} \right)^{a_3 + \frac{1}{r'}} \cdot |I|.$$

Because we want some decay with respect to $\frac{\text{dist}(H', I_0)}{|I_0|}$, we estimate the $\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}$ in several ways:

$$\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}} \lesssim \widetilde{\widetilde{\text{size}_{I_0} \mathbf{1}_{H'}}}, \quad \lesssim 2^{-d\lambda}, \quad \lesssim 2^{-m\lambda}.$$

Let $0 \leq \beta_1, \beta_2, \beta_3 \leq 1$, with $\beta_1 + \beta_2 + \beta_3 = 1$.

$$\left| \sum_k \Lambda_{\tilde{\Gamma}_{\text{loc}}^\alpha}(f_k, g_k, h_k) \right| \lesssim \left(\widetilde{\text{size}_I \mathbf{1}_F} \right)^{a_1 + \frac{1}{r_1} - \nu_1} \cdot \left(\widetilde{\text{size}_I \mathbf{1}_G} \right)^{a_2 + \frac{1}{r_2} - \nu_2} \cdot \left(\widetilde{\widetilde{\text{size}_I \mathbf{1}_{H'}}} \right)^{\beta_1(a_3 + \frac{1}{r'})}. \quad (5.5)$$

$$2^{-m\beta_2(a_3 + \frac{1}{r'})} \cdot \sum_{n_1, n_2, n_3} 2^{-n_1(\nu_1 - \theta_1)} 2^{-n_2(\nu_2 - \theta_2)} 2^{-n_3[\beta_3(a_3 + \frac{1}{r'}) - \theta_3]} |F|_1^\theta |G|^{\theta_2} \quad (5.6)$$

In the end, we will have $\nu_1 \sim \frac{1}{s_1}, \nu_2 \sim \frac{1}{s_2}$. For (5.6) to converge, we must have

$$\frac{1}{s} + \beta_3 \left(a_3 + \frac{1}{r'} \right) > 1. \quad (5.7)$$

Then the whole expression adds up to

$$2^{-d(\lambda[\beta_3(a_3 + \frac{1}{r'}) - \theta_3] - (\nu_1 - \theta_1) - (\nu_2 - \theta_2))} |F|^{\nu_1} |G|^{\nu_2},$$

so we need

$$\lambda \left[\beta_3 \left(a_3 + \frac{1}{r'} \right) - \theta_3 \right] - (\nu_1 - \theta_1) - (\nu_2 - \theta_2) > 0. \quad (5.8)$$

Conditions 5.7 and 5.8 can be summarized as

$$-\lambda\beta_3\left(a_3 + \frac{1}{r'}\right) < \frac{1}{s'} < \beta_3\left(a_3 + \frac{1}{r'}\right).$$

Note that $\mathfrak{h} = \lambda\beta_2\left(a_3 + \frac{1}{s'}\right)$. In order to establish vector-valued estimates for the generalized paraproducts for iterated vector spaces, we will want $\mathfrak{h}s' > 1$. For the exponents of the sizes, we have

$$\alpha_1 = a_1 + \frac{1}{r_1} - \frac{1}{s_1} - \epsilon, \quad \alpha_2 = a_2 + \frac{1}{r_2} - \frac{1}{s_2} - \epsilon, \quad \alpha_3 = \beta_1\left(a_3 + \frac{1}{r'}\right).$$

Before we go on, we remark that $\alpha_1 + \frac{1}{s_1} \sim 1$ and $\alpha_2 + \frac{1}{s_2} \sim 1$. □

Theorem 25. For $(r, s) \in \mathcal{I}_2(\alpha)$, and $(p, q, t) \in \mathcal{D}_{r,s}(\alpha)$,

$$\|\tilde{\Pi}^\alpha(f_{kl}, g_{kl})\|_{L^t(I^s(I^r))} \lesssim \|f_{kl}\|_{L^p(I^{s_1}(I^{r_1}))} \|g_{kl}\|_{L^q(I^{s_2}(I^{r_2}))}.$$

Proof. To establish this result, we need to estimate

$$\sum_{n_1, n_2, n_3} \sum_{I_0 \in \mathcal{I}^{n_1, n_2, n_3}} \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F\right)^{\alpha_1} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G\right)^{\alpha_2} \cdot \left(\widetilde{\widetilde{\text{size}}}_{I_0} \mathbf{1}_{H'}\right)^{\alpha_3}. \quad (5.9)$$

$$\frac{\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\|_{s_1}}{|I_0|^{1/s_1}} \cdot \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_{I_0}\|_{s_2}}{|I_0|^{1/s_2}} \cdot \frac{\|\mathbf{1}_{H'} \cdot \phi_{I_0}\|_{s'}}{|I_0|^{1/s'}} \quad (5.10)$$

There are two types of averages of $\mathbf{1}_{H'}$ here:

$$\sup_{I \subseteq I_0} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\lambda}} dx \quad \text{and} \quad (5.11)$$

$$\frac{1}{|I'|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I')}{|I'|}\right)^{\mathfrak{h}s'}} dx. \quad (5.12)$$

We let $1 + \tilde{\lambda} := \min 1 + \lambda, \mathfrak{h}s'$, and the new size we need to deal with is

$$\sup_{I \subseteq I_0} \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\tilde{\lambda}}} dx.$$

The collection \mathcal{J}_{n_3} of intervals is so that

$$2^{-n_3-1} \leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{H'} \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{(1+\tilde{\lambda})}} dx \leq 2^{-n_3} \lesssim 2^{-d\tilde{\lambda}}.$$

Using this stopping time and previous estimates, we have

$$\begin{aligned} & \left| \sum_l \sum_k \sum_I \frac{1}{|I|^{1/2}} \langle f_{kl}, \phi_I \rangle \langle g_{kl}, \phi_I \rangle \langle h_{kl}, \varphi_I^{3,\alpha} \rangle \right| \lesssim \\ & \lesssim \sum_{n_1, n_2, n_3} 2^{-n_1(\nu_1 - \theta_1)} 2^{-n_2(\nu_2 - \theta_2)} 2^{-n_3(\alpha_3 + \frac{1}{s'} - \theta_3)} |F|^{\theta_1} |G|^{\theta_3} \lesssim \\ & \lesssim 2^{-d[\tilde{\lambda}(\alpha_3 + \frac{1}{s'} - \theta_3) - (\nu_1 - \theta_1) - (\nu_2 - \theta_2)]} |F|^{\nu_1} |G|^{\nu_2}. \end{aligned}$$

The restrictions are

$$\frac{1}{t} + \alpha_3 + \frac{1}{s'} > 1 \quad \text{and} \quad 1 + \tilde{\lambda} \left(\alpha_3 + \frac{1}{s'} \right) > \frac{1}{t}$$

or, equivalently,

$$-\tilde{\lambda} \left(\alpha_3 + \frac{1}{s'} \right) < \frac{1}{t'} < \alpha_3 + \frac{1}{s'}. \quad (5.13)$$

□

Iterated l^1 spaces

Later on, in section 6.3, we will need estimates for $\tilde{\Pi}^\alpha$ in $L^s(l^1(l^1(\dots l^1(l^1))))$. This case will be easier to deal with, because it doesn't involve the decay of φ^α .

Theorem 26. *For any $n \geq 1$, $(\underbrace{1, \dots, 1}_n) \in \mathcal{J}_n(\alpha)$, and $\mathcal{D}_{(1, \dots, 1)}(\alpha) = \text{Range}(\tilde{\Pi}^\alpha)$. In particular, for any $1 < p, q < \infty$, $\frac{1}{1+\alpha} < s < \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$,*

$$\left\| \sum_{k_1, k_2, \dots, k_n} \tilde{\Pi}^\alpha(f_{k_1, k_2, \dots, k_n}, g_{k_1, k_2, \dots, k_n}) \right\|_s \lesssim \left\| \left(\sum_{k_1, k_2, \dots, k_n} |f_{k_1, k_2, \dots, k_n}|^2 \right)^{1/2} \right\|_p \cdot \left\| \left(\sum_{k_1, k_2, \dots, k_n} |g_{k_1, k_2, \dots, k_n}|^2 \right)^{1/2} \right\|_q.$$

Proof. The proof goes by induction; knowing that $\underbrace{(1, \dots, 1)}_n \in \mathcal{J}_n(\alpha)$, we prove that $\underbrace{(1, \dots, 1)}_{n+1} \in \mathcal{J}_{n+1}(\alpha)$. We illustrate this procedure in the case $n = 1$. We already proved in section 5.0.4 that whenever $\frac{1}{r_1} + \frac{1}{r_2} = 1$,

$$\left\| \sum_k |\tilde{\Pi}^\alpha(f_k, g_k)| \right\|_s \lesssim \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_p \cdot \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_q$$

We need a localized variant, similar to Lemma 9, namely

$$\begin{aligned} \left\| \sum_k |\tilde{\Pi}_{\text{loc}}^\alpha(f_k, g_k)| \right\|_1 &\lesssim (\widetilde{\text{size}}_{I_0} \mathbf{1}_F)^{\alpha_1} \cdot (\widetilde{\text{size}}_{I_0} \mathbf{1}_G)^{\alpha_2} \cdot (\widetilde{\text{size}}_{I_0} \mathbf{1}_{H'})^{\alpha_3} \cdot \\ &\quad \cdot \left\| \left(\sum_k |f_k|^{r_1} \right)^{1/r_1} \right\|_{\tilde{\chi}_{I_0}} \cdot \left\| \left(\sum_k |g_k|^{r_2} \right)^{1/r_2} \right\|_{\tilde{\chi}_{I_0}}. \end{aligned}$$

We recall that, given a fixed interval I_0 ,

$$\tilde{\Pi}_{\text{loc}}^\alpha(f, g)(x) := \sum_{I \subseteq I_0} \frac{1}{|I|^{1/2}} \langle f \cdot \mathbf{1}_F, \psi_I \rangle \langle g \cdot \mathbf{1}_G, \psi_I \rangle \varphi_I^\alpha(x) \cdot \mathbf{1}_{H'}(x).$$

When the target space is L^1 , we can't obtain estimates without interpolating with quasi-Banach spaces. For this reason, the proofs require a little bit more work, but it is standard: E_1, E_2, E_3 are sets of finite measure, the exceptional set Ω corresponds to large values of $\mathcal{M}\mathbf{1}_{E_1}$ and $\mathcal{M}\mathbf{1}_{E_2}$. We dualize the above expression by some $h = \{h_k(x)\}_k$, with $\sup_k |h_k(x)| \leq \mathbf{1}_{E'_3}(x)$, where $E'_3 := E_3 \setminus \Omega$.

Assume that all the intervals are so that $\frac{\text{dist}(I, \Omega^c)}{|I|} \sim 2^d$, and the stopping time is performed as before. The only difference is that, for \mathcal{J}_{n_3} , we demand just

$$\begin{aligned} 2^{-n_3-1} &\leq \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E'_3} \cdot \frac{1}{\left(1 + \frac{\text{dist}(x, I)}{|I|}\right)^{1+\alpha}} dx \leq 2^{-n_3} \lesssim 2^{-d\alpha}. \\ \left| \sum_k \Lambda_{\tilde{\Pi}_{\text{loc}}^\alpha}(f_k, g_k, h_k) \right| &\lesssim \sum_{n_1, n_2, n_3} \sum_{\substack{I \in \mathcal{J}_{n_1, n_2, n_3} \\ I \subseteq I_0}} (\widetilde{\text{size}}_I \mathbf{1}_F)^{a_1} \cdot (\widetilde{\text{size}}_I \mathbf{1}_G)^{a_2} \cdot (\widetilde{\text{size}}_I \mathbf{1}_{H'})^{a_3} \cdot \\ &\quad \cdot \frac{\|\mathbf{1}_F \cdot \tilde{\chi}_I\|_{r_1}}{|I|^{1/r_1}} \frac{\|\mathbf{1}_G \cdot \tilde{\chi}_I\|_{r_2}}{|I|^{1/r_2}} \cdot |I| \lesssim \\ &\lesssim \sum_{n_1, n_2, n_3} \sum_{\substack{I \in \mathcal{J}_{n_1, n_2, n_3} \\ I \subseteq I_0}} (\widetilde{\text{size}}_I \mathbf{1}_F)^{a_1 + \frac{1}{r_1}} \cdot (\widetilde{\text{size}}_I \mathbf{1}_G)^{a_2 + \frac{1}{r_2}} \cdot (\widetilde{\text{size}}_I \mathbf{1}_{H'})^{a_3} \cdot |I|. \end{aligned}$$

Here $a_1 + \frac{1}{r_1} \sim 1$, $a_2 + \frac{1}{r_2} \sim 1$, and $a_3 \sim 1 - \epsilon$. The functions ψ_I , are Schwartz, so we can assume the 1_{E_1} s and 1_{E_2} are mainly supported on $5I_0$. In the end, we get

$$\begin{aligned}
& \left| \sum_k \Lambda_{\tilde{\Pi}_{\text{loc}}^\alpha}(f_k, g_k, h_k) \right| \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{1-\nu_1-\epsilon} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{1-\nu_2-\epsilon} \cdot \left(\widetilde{\widetilde{\text{size}}}_{I_0} \mathbf{1}_{H'} \right)^{1-\nu_2-\epsilon} \cdot \\
& \cdot \sum_{n_1, n_2, n_3} 2^{-n_1 \nu_1} 2^{-n_2 \nu_2} 2^{-n_3 \nu_3} 2^{n_1 \theta_1} 2^{n_2 \theta_2} 2^{n_3 \theta_3} |E_1|^{\theta_1} |E_2|^{\theta_2} |E_3|^{\theta_3} \lesssim \\
& \lesssim \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_F \right)^{1-\nu_1-\epsilon} \cdot \left(\widetilde{\text{size}}_{I_0} \mathbf{1}_G \right)^{1-\nu_2-\epsilon} \cdot \left(\widetilde{\widetilde{\text{size}}}_{I_0} \mathbf{1}_{H'} \right)^{1-\nu_2-\epsilon} \cdot \\
& \cdot \left(2^d \frac{|E_1|}{|E_3|} \right)^{\nu_1-\theta_1} \left(2^d \frac{|E_2|}{|E_3|} \right)^{\nu_2-\theta_2} 2^{-d\alpha(\nu_3-\theta_3)} |E_1|^{\theta_1} |E_2|^{\theta_2} |E_3|^{\theta_3}
\end{aligned}$$

From here, we conclude the desired result. The conditions are that $\nu_1 + \nu_2 + \nu_3 > 1$, and $(\nu_1 - \theta_1) + (\nu_2 - \theta_2) < \alpha(\nu_3 - \theta_3)$. These can easily be satisfied, since we will have $\nu_1 \sim \frac{1}{s_1}$, $\nu_2 \sim \frac{1}{s_2}$, and $\frac{1}{s_1} + \frac{1}{s_2} = 2$. Once we have this localization lemma, we can deduce both the vector-valued result and a localized vector-valued variant, the second one being necessary for the proof of the $n = 3$ case. \square

CHAPTER 6

APPLICATIONS

6.1 A Rubio de Francia Theorem for Iterated Fourier Integrals and the initial AKNS problem

Now we return to the operator that motivated the study of vector-valued bilinear operators. Recall that

$$T_r(f, g)(x) = \left(\sum_{k=1}^N \left| \int_{a_k < \xi_1 < \xi_2 < b_k} \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^r \right)^{\frac{1}{r}}.$$

We prove the boundedness of T_r within some range \mathcal{D}_r , and then apply this to answer the question regarding the boundedness of M from (1.5).

Theorem 27. *If $1 \leq r \leq 2$, then*

$$\|T_r(f, g)\|_s \lesssim \|f\|_p \|g\|_q$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, and $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'}\right)$ are in the convex cover illustrated in

$$\text{Co}\left((0, 0, 1), \left(\frac{1}{2} + \frac{1}{r'}, \frac{1}{2}, -\frac{1}{r'}\right), \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{r'}, -\frac{1}{r'}\right), \left(\frac{1}{2} + \frac{1}{r'}, 0, \frac{1}{2} - \frac{1}{r'}\right), \left(0, \frac{1}{2} + \frac{1}{r'}, \frac{1}{2} - \frac{1}{r'}\right)\right).$$

On the other hand, if $r \geq 2$, T_r is a bounded operator with the same range as the BHT operator.

Remark 15. *For the RF_v operator, the relation between the exponent v and the L^p spaces is given by $\frac{1}{v} + \frac{1}{p} < 1$. In the bilinear case, for the operator T_r , the constraint that appears naturally is $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} < 2$. The necessity of this condition can be seen through a counterexample similar to that for RF_v from [9].*

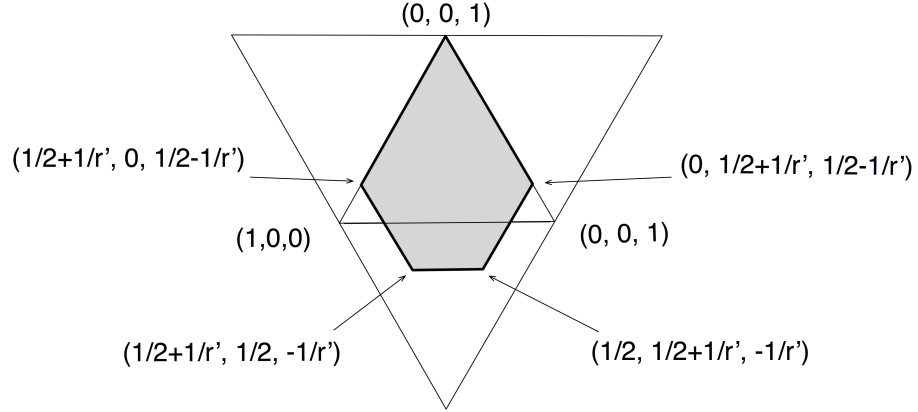


Figure 6.1: Range for T_r operator for $1 \leq r \leq 2$

Using Rubio de Francia's theorem, the proof of the boundedness of the operator T_r follows from

$$\left\| \left(\sum_{k=1}^N |BHT(f_k, g_k)|^r \right)^{1/r} \right\|_s \lesssim \left\| \left(\sum_{k=1}^N |f_k|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_{k=1}^N |g_k|^{r_2} \right)^{1/r_2} \right\|_q,$$

where $r_1, r_2 > 1$ will be chosen later so that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$.

Returning to the initial question that motivated the study of vector-valued BHT , its proof follows from Theorem 20, with r_1, r_2 chosen carefully so that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$.

Proof of Theorem 27. We start with the case $r \geq 2$; this follows from

$$\left\| \left(\sum_k |BHT(P_{I_k} f, P_{I_k} g)(x)|^2 \right)^{1/2} \right\|_s \lesssim \left\| \left(\sum_k |P_{I_k} f|^{r_1} \right)^{1/r_1} \right\|_p \left\| \left(\sum_k |P_{I_k} g|^{r_2} \right)^{1/r_2} \right\|_q, \quad (6.1)$$

for any $1 < p, q < \infty, \frac{2}{3} < s < \infty$.

This is implied by Rubio de Francia's theorem, if one can find r_1 and r_2 with

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2} \text{ and}$$

$$\frac{1}{p} < \frac{1}{r'_1}, \quad \frac{1}{q} < \frac{1}{r'_2}.$$

This is possible as long as $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < \frac{1}{r'_1} + \frac{1}{r'_2} = \frac{3}{2}$, which coincides with the condition we have for the range of *BHT*.

The case $1 \leq r < 2$ is similar; for p, q , and s as above, one needs to find r_1 and $r_2 \geq 2$ so that

$$2 - \frac{1}{r} = \frac{1}{r'_1} + \frac{1}{r'_2} > \frac{1}{p} + \frac{1}{q}.$$

Note that $\frac{1}{p} < \frac{1}{r'_1} = 1 - \frac{1}{r} + \frac{1}{r_2} \leq \frac{1}{r'} + \frac{1}{2}$, and similarly for q . Because of this restriction, the operator T_r is bounded as long as admissible triple $(\frac{1}{p}, \frac{1}{q}, \frac{1}{s'})$ is in the convex hull of the points

$$(0, 0, 1), \left(\frac{1}{2} + \frac{1}{r'}, \frac{1}{2}, -\frac{1}{r'}\right), \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{r'}, -\frac{1}{r'}\right), \left(\frac{1}{2} + \frac{1}{r'}, 0, \frac{1}{2} - \frac{1}{r'}\right), \left(0, \frac{1}{2} + \frac{1}{r'}, \frac{1}{2} - \frac{1}{r'}\right).$$

□

Remark 16. *An alternative way of proving the boundedness of T_r within the range mentioned in 27 is by interpolating between*

$$L^{p_1} \times L^{q_1} \rightarrow L^{s_1}(l^2) \quad \text{with } p_1, q_1, s_1 \text{ in the range of the BHT operator and} \quad (6.2)$$

$$L^{p_2} \times L^{q_2} \rightarrow L^{s_2}(l^1) \quad \text{with } p_2, q_2, s_2 \geq 1. \quad (6.3)$$

Both estimates 6.2 and 6.3 follow from the vector-valued estimates for *BHT* of [14], even if the case $r = 1$ is not transparent in the referenced paper.

6.1.1 Boundedness of operators M_1 and M_2

In what follows we prove the boundedness of operators M_1 and M_2 presented in 1.6 and 1.7:

$$M_1(f_1, f_2, g)(\xi) = \sum_{\omega} \int_{\substack{x_1 < x_2, x_1, x_2 \in \omega_L \\ x_3 \in \omega_R}} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3$$

and

$$M_2(f_1, f_2, g)(\xi) = \sum_{\omega} \int_{\substack{x_1 < L(\omega_L), x_2 \in \omega_L \\ x_3 \in \omega_R}} \hat{f}_1(x_1) \hat{f}_2(x_2) g(x_3) e^{2\pi i \xi(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3$$

For both operators, we are going to use the triangle inequality in L^r , the target space for operators M_1 and M_2 . However, if $r < 1$ this inequality is not available anymore for the quasi-norm $\|\cdot\|_r$ and instead we use the triangle inequality for $\|\cdot\|_r^r$. This is the only difference between the Banach and quasi-Banach case, and for simplicity we assume $r \geq 1$. Also, as previously stated, we assume $\|g\|_p = 1$.

Proposition 18. *Let $1 < p < 2$ and $\frac{1}{r} = \frac{1}{s} + \frac{1}{p'} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'}$. Then*

$$\|M_1(f_1, f_2, g)\|_r \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2} \|g\|_p.$$

Proof. Recall that $\omega \in \mathcal{D}$ is the mesh of dyadic intervals contained in $[0, 1]$, and we identify them with their preimage: $\omega \sim \varphi^{-1}(\omega)$. We rewrite M_1 as

$$M_1(f_1, f_2, g)(\xi) = \sum_{\omega} BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)(\xi) \cdot g \cdot \widehat{\mathbf{1}_{\omega_R}}(\xi)$$

Then

$$\begin{aligned}
\|M_1(f_1, f_2, g)\|_r &\lesssim \sum_{k \geq 0} \left\| \sum_{|\omega|=2^{-k}} BHT(P_{\omega_L} f_1, P_{\omega_L} f_2) \cdot \widehat{g \cdot \mathbf{1}_{\omega_R}} \right\|_r \lesssim \\
&\lesssim \sum_{k \geq 0} \left\| \left(\sum_{|\omega|=2^{-k}} |BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)|^p \right)^{1/p} \cdot \left(\sum_{|\omega|=2^{-k}} |\widehat{g \cdot \mathbf{1}_{\omega_R}}|^{p'} \right)^{1/p'} \right\|_r \lesssim \\
&\lesssim \sum_{k \geq 0} \left\| \left(\sum_{|\omega|=2^{-k}} |BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)|^p \right)^{1/p} \right\|_s \cdot \left(\sum_{|\omega|=2^{-k}} \|\widehat{g \cdot \mathbf{1}_{\omega_R}}\|_{p'}^{p'} \right)^{1/p'}
\end{aligned}$$

We estimate $\|\widehat{g \cdot \mathbf{1}_{\omega_R}}\|_{p'} \lesssim \|g \cdot \mathbf{1}_{\omega_R}\|_p = 2^{-\frac{k}{p}}$ using the Hausdorff-Young theorem.

Also, there are 2^k dyadic intervals of length 2^{-k} in $[0, 1]$ and because of this

$$\|M_1(f_1, f_2, g)\|_r \lesssim \sum_{k \geq 0} 2^{-k\left(\frac{1}{p} - \frac{1}{p'}\right)} \left\| \left(\sum_{|\omega|=2^{-k}} |BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)|^p \right)^{1/p} \right\|_s.$$

If we estimate the last term using the operator T_p directly, we will not obtain the full range stated above, as there will appear extra constraints of the type

$$\frac{1}{p_1} + \frac{1}{p} < \frac{3}{2}, \quad \frac{1}{p_2} + \frac{1}{p} < \frac{3}{2}.$$

Instead, using Hölder and the fact that $1 < p < 2$, we have

$$\|BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)\|_{l^p(\omega)} \leq \|BHT(P_{\omega_L} f_1, P_{\omega_L} f_2)\|_{l^2(\omega)} \cdot 2^{k\left(\frac{1}{p} - \frac{1}{2}\right)}.$$

Using the boundedness of T_2 , we have $\|M_1(f_1, f_2, g)\|_r \lesssim \sum_{k \geq 0} 2^{-k\left(\frac{1}{2} - \frac{1}{p'}\right)} \|f_1\|_{p_1} \|f_2\|_{p_2}$.

□

Proposition 19. Let $1 < p < 2$ and $\frac{1}{r} = \frac{1}{s} + \frac{1}{p'} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p'}$. Then

$$\|M_2(f_1, f_2, g)\|_r \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2} \|g\|_p,$$

provided $\frac{1}{p_2} + \frac{1}{p'} < 1$.

Proof. First, we remark that

$$|M_2(f_1, f_2, g)(\xi)| \leq \sum_{\omega} |Cf_1(\xi)| \cdot |P_{\omega_L} f_2(\xi)| \cdot |\widehat{g \cdot \omega_R}(\xi)|,$$

where C is the Carleson operator, bounded on L^p whenever $1 < p < \infty$. From here on, the estimates are similar to those in proposition 18, but instead of the bilinear operator $T_r(f, g)$ we will have to use the more restrictive Rubio de Francia operator RF_ν :

$$\begin{aligned}
\|M_2(f_1, f_2, g)\|_r &\leq \sum_{k \geq 0} \|Cf_1 \cdot \left(\sum_{|\omega|=2^{-k}} |P_{\omega_L} f_2|^p \right)^{1/p} \cdot \left(\sum_{|\omega|=2^{-k}} |g \cdot \widehat{\mathbf{1}_{\omega_R}}|^{p'} \right)^{1/p'}\|_r \leq \\
&\leq \sum_{k \geq 0} \|Cf_1\|_{p_1} \cdot \left\| \left(\sum_{|\omega|=2^{-k}} |P_{\omega_L} f_2|^p \right)^{1/p} \right\|_{p_2} \cdot \left\| \left(\sum_{|\omega|=2^{-k}} \|g \cdot \widehat{\mathbf{1}_{\omega_R}}\|_{p'}^{p'} \right)^{1/p'} \right\|_{p_2} \leq \\
&\leq \sum_{k \geq 0} 2^{k(\frac{1}{p} - \frac{1}{\nu})} \|Cf_1\|_{p_1} \cdot \left\| \left(\sum_{|\omega|=2^{-k}} |P_{\omega_L} f_2|^\nu \right)^{1/\nu} \right\|_{p_2} \cdot \left\| \left(\sum_{|\omega|=2^{-k}} \|g \cdot \widehat{\mathbf{1}_{\omega_R}}\|_{p'}^{p'} \right)^{1/p'} \right\|_{p_2} \leq \\
&\leq \sum_{k \geq 0} 2^{-k(\frac{1}{\nu} - \frac{1}{p'})} \|f_1\|_{p_1} \|RF_\nu(f_2)\|_{p_2}.
\end{aligned}$$

If $p_2 \geq 2$, we can take $\nu = 2$ and there are no other restrictions. In the case $p_2 < 2$, Rubio de Francia requires $\frac{1}{\nu} + \frac{1}{p_2} < 1$. This and the condition $\frac{1}{\nu} - \frac{1}{p'} > 0$ (so that the geometric series above is finite) can be summarized as $\frac{1}{p_2} + \frac{1}{p'} < 1$. \square

6.2 Tensor Products of Bilinear Operators

Iterating the proof and results of theorem 20, we can prove the following more general result:

Theorem 28. *The Bilinear Hilbert Transform satisfies the following vector-valued estimates:*

$$BHT : L^p(l^1(l^2(\dots(l^K)\dots))) \times L^q(l^2(l^2(\dots(l^K)\dots))) \mapsto L^r(l^1(l^2(\dots(l^K)\dots)))$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with $1 < p, q \leq \infty$ and $2/3 < r < \infty$, and for every $1 \leq j \leq K$, $\frac{1}{r_1^j} + \frac{1}{r_2^j} = \frac{1}{r^j}$, $r_1^j, r_2^j, (r^j)' \geq 2$.

Remark 17. *In fact, the same results hold for iterates of mixed L^k and l^j spaces. The proof is identical.*

Using the above Theorem 28, we will prove the boundedness of the tensor product

$$BHT \otimes \underbrace{\Pi \otimes \dots \otimes \Pi}_n : L^p(\mathbb{R}^{n+1}) \times L^q(\mathbb{R}^{m+1}) \rightarrow L^r(\mathbb{R}^{n+1}), \text{ and}$$

$$\Pi \otimes \dots \otimes \Pi \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi : L^p(\mathbb{R}^{n+1}) \times L^q(\mathbb{R}^{m+1}) \rightarrow L^r(\mathbb{R}^{n+1}).$$

whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, with $\frac{2}{3} < r < \infty$, $1 \leq p, q < \infty$.

Results about tensor products of bilinear operators that were previously known are:

1. $\Pi \otimes \Pi : L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \rightarrow L^s(\mathbb{R}^2)$ whenever $1 < p, q \leq \infty$, $\frac{1}{2} < s < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.
2. The same is true for $\underbrace{\Pi \otimes \dots \otimes \Pi}_{n \text{ times}} : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^s(\mathbb{R}^n)$.
3. $BHT \otimes BHT$ doesn't satisfy any L^p estimates.
4. $BHT \otimes \Pi : L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \rightarrow L^s(\mathbb{R}^2)$ whenever $1 < p, q \leq \infty$, $\frac{1}{2} < s < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, and in addition

$$\frac{1}{p} + \frac{2}{q} < 2, \quad \text{and} \quad \frac{1}{q} + \frac{2}{p} < 2.$$

The first three results were proved in [26], while the last one is from [14]. Our result is a generalization of the previously known estimate for $BHT \otimes \Pi$, in the sense that the range is extended to get $Range(BHT)$. Moreover, instead of a paraproduct, we can have arbitrarily many $\Pi \otimes \dots \otimes \Pi$, and even for this operator the range is still $Range(BHT)$.

Remark 18. *The methods of the proof and our approach for tensor products allow us to give another proof for the boundedness of $\Pi \otimes \dots \otimes \Pi$, within the same range as $\text{Range}(\Pi)$.*

If $T_1 : L^p(\mathbb{R}^{n_1}) \times L^q(\mathbb{R}^{n_1}) \rightarrow L^r(\mathbb{R})$ and $T_2 : L^p(\mathbb{R}^{n_2}) \times L^q(\mathbb{R}^{n_2}) \rightarrow L^r(\mathbb{R})$ are two bilinear operators, then the tensor product

$$T_1 \otimes T_2 : L^p(\mathbb{R}^{n_1+n_2}) \times L^q(\mathbb{R}^{n_1+n_2}) \rightarrow L^r(\mathbb{R})$$

will act as T_1 in the first variable and as T_2 in the second variable. In our case, the operators are given by singular multipliers, and in this case we can give a characterization of the tensor product. Assume

$$T_1(f, g)(x) = \int_{\mathbb{R}^{2n_1}} \hat{f}(\xi_1) \hat{g}(\xi_2) m_1(\xi_1, \xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2$$

and similarly

$$T_2(f, g)(y) = \int_{\mathbb{R}^{2n_2}} \hat{f}(\eta_1) \hat{g}(\eta_2) m_2(\eta_1, \eta_2) e^{2\pi i y \cdot (\eta_1 + \eta_2)} d\eta_1 d\eta_2.$$

Then the multiplier for the tensor product is precisely $m_1(\xi_1, \xi_2) \cdot m_2(\eta_1, \eta_2)$:

$$T_1 \otimes T_2(f, g)(x, y) = \int_{\mathbb{R}^{2n}} \hat{f}(\xi_1, \eta_1) \hat{g}(\xi_2, \eta_2) m_1(\xi_1, \xi_2) m_2(\eta_1, \eta_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} e^{2\pi i y \cdot (\eta_1 + \eta_2)} d\xi_1 d\xi_2 d\eta_1 d\eta_2.$$

Recall that the multiplier associated with BHT is $\text{sgn}(\xi_1 - \xi_2)$, while the multiplier for a paraproduct is a classical Marcinkiewicz-Mikhlin-Hörmander multiplier $m(\xi_1, \xi_2)$, smooth away from the origin, satisfying the condition $|\partial^\alpha(\xi)| \lesssim |\xi|^{-|\alpha|}$ for sufficiently many multi-indices α . The decay in m allows one to approximate the multiplier by a finite number of sums of the form

$$\sum_k \hat{\varphi}_k(\xi_1) \hat{\psi}_k(\xi_2) \hat{\psi}_k(\xi_1 + \xi_2) \quad \text{or} \quad \sum_k \hat{\psi}_k(\xi_1) \hat{\psi}_k(\xi_2) \hat{\varphi}_k(\xi_1 + \xi_2).$$

With the purpose of simplifying the notation, we denote by P_k the Littlewood-Paley projection onto $\{|\xi| \sim 2^k\}$ (which is really the convolution with $\psi_k(\cdot)$), and by Q_k the projection onto $\{|\xi| \leq 2^k\}$, corresponding to the convolution with φ_k . Then we can regard paraproducts as being expressions of the form

$$\Pi(f, g)(x) = \sum_k P_k(Q_k f \cdot P_k g)(x, y) \quad \text{or} \quad (6.4)$$

$$\Pi(f, g)(x) = \sum_k Q_k(P_k f \cdot P_k g)(x, y). \quad (6.5)$$

Proposition 20. *Let $T_m : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$ be a bilinear operator with smooth symbol m , and $\Pi : L^p(\mathbb{R}) \times L^q(\mathbb{R}) \rightarrow L^r(\mathbb{R})$ a paraproduct as described above.*

(a) *If Π is given by 6.4, then*

$$(T_m \otimes \Pi)(f, g)(x, y) = \sum_k P_k^2(T_m(Q_k^y f, P_k^y g)(x)).$$

(b) *If Π is given by 6.5, then*

$$(T_m \otimes \Pi)(f, g)(x, y) = \sum_k Q_k^2(T_m(P_k^y f, P_k^y g)(x)) = \sum_k T_m(P_k^y f, P_k^y g)(x).$$

Here we need to explain the notation: P_k^2 denotes the projection onto $|\xi_2| \sim 2^k$ in the second variable, and $Q_k^y f$ is a function of x only, with the variable y fixed. The exact formulas are

$$\begin{aligned} Q_k^y f(x) &= \int_{\mathbb{R}} \varphi_k(s) f(x, y - s) ds, & Q_k^2 f(x, y) &= \int_{\mathbb{R}} \varphi_k(s) f(x, y - s) ds, \\ P_k^y f(x) &= \int_{\mathbb{R}} \psi_k(s) f(x, y - s) ds, & P_k^2 f(x, y) &= \int_{\mathbb{R}} \psi_k(s) f(x, y - s) ds. \end{aligned}$$

Proof. This is going to be a series of direct computations:

$$\begin{aligned}
& (T_m \otimes \Pi)(f, g)(x, y) = \\
& = \int_{\mathbb{R}^{2n+2}} \hat{f}(\xi_1, \eta_1) \hat{g}(\xi_2, \eta_2) m(\xi_1, \xi_2) \left(\sum_k \hat{\varphi}_k(\eta_1) \hat{\psi}_k(\eta_2) \hat{\psi}_k(\eta_1 + \eta_2) \right) e^{2\pi i x \cdot (\xi_1 + \xi_2)} e^{2\pi i y \cdot (\eta_1 + \eta_2)} d\xi d\eta = \\
& = \sum_k \int_{\mathbb{R}^{2n+2}} \hat{f}(\xi_1, \eta_1) \hat{g}(\xi_2, \eta_2) m(\xi_1, \xi_2) \hat{\varphi}_k(\eta_1) \hat{\psi}_k(\eta_2) \left(\int_{\mathbb{R}} \psi_k(s) e^{-2\pi i s(\eta_1 + \eta_2)} ds \right) \cdot \\
& \cdot e^{2\pi i x \cdot (\xi_1 + \xi_2)} e^{2\pi i y \cdot (\eta_1 + \eta_2)} d\xi d\eta = \sum_k \int_{\mathbb{R}} \psi_k(s) (T_m(Q_k^{y-s} f, P_k^{y-s} g)(x)) ds = \\
& = \sum_k P_k^2 T_m(Q_k^y f, P_k^y g)(x).
\end{aligned}$$

□

A final ingredient that we will need in the proof of theorem 29 is the following lemma which appears in [27] and [11]:

Lemma 10. *Let $f \in \mathcal{S}(\mathbb{R}^n)$, and $1 \leq l \leq n$, and $\{i_1, \dots, i_l\} \subset \{1, \dots, n\}$. Then*

$$\|f\|_{L^p} \lesssim \left\| \left(\sum_{k_1, \dots, k_l} |P_{k_1}^{i_1} \dots P_{k_l}^{i_l} f|^2 \right)^{1/2} \right\|_{L^p}$$

for any $0 < p < \infty$.

Lemma 10 above states that the L^p norm of f is bounded by the L^p norm of a square function associated with the variables x_{i_1}, \dots, x_{i_l} , even when $0 < p \leq 1$. In the case $p > 1$, it is well known that the two norms are equivalent. When $p < 1$, the proof makes use of multi-parameter Hardy spaces.

Theorem 29. *For any p, q, r with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with $1 < p, q \leq \infty$ and $2/3 < r < \infty$:*

$$\|BHT \otimes \Pi \otimes \dots \otimes \Pi(f, g)\|_{L^r(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})} \|g\|_{L^q(\mathbb{R}^{n+1})} \quad \text{and}$$

$$\|\Pi \otimes \dots \otimes \Pi \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi(f, g)\|_{L^r(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n+1})} \|g\|_{L^q(\mathbb{R}^{n+1})}.$$

Proof. We partition $\{1, \dots, n\}$ into three subsets of indices $\mathcal{I}_1, \mathcal{I}_2$ and \mathcal{I}_3 so that

- if $k \in \mathcal{I}_1$, then $\Pi(f, g)(y) = \sum_k P_k(Q_k f \cdot P_k g)(y)$
- if $k \in \mathcal{I}_2$, then $\Pi(f, g)(y) = \sum_k P_k(P_k f \cdot Q_k g)(y)$ and
- if $k \in \mathcal{I}_3$, then $\Pi(f, g)(y) = \sum_k Q_k(P_k f \cdot P_k g)(y)$.

Because the projections onto different coordinates commute, i.e. $P_k^i P_l^j = P_l^j P_k^i$ and $Q_k^i P_l^j = P_l^j Q_k^i$, we can assume

$$\mathcal{I}_1 = \{1, \dots, l\}, \quad \mathcal{I}_2 = \{l+1, \dots, l+d\}, \quad \mathcal{I}_3 = \{l+d+1, \dots, n\}.$$

Of course, we allow the possibility that one or even two of these sets of indices are empty. With this assumption, we have from Proposition 20 applied iteratively,

$$\begin{aligned} BHT \otimes \Pi \otimes \dots \Pi(f, g)(x, y_1, \dots, y_n) &= \\ &= \sum_{k_1, \dots, k_n} P_{k_1}^1 \dots P_{k_l}^l P_{k_{l+1}}^{l+1} \dots P_{k_{l+d}}^{l+d} Q_{k_{l+d+1}}^{l+d+1} \dots Q_{k_n}^n \circ \\ &BHT(Q_{k_1}^{y_1} \dots Q_{k_l}^{y_l} P_{k_{l+1}}^{y_{l+1}} \dots P_{k_n}^{y_n} f, P_{k_1}^{y_1} \dots P_{k_l}^{y_l} Q_{k_{l+1}}^{y_{l+1}} \dots Q_{k_{l+d}}^{y_{l+d}} P_{k_{l+d+1}}^{y_{l+d+1}} \dots P_{k_n}^{y_n} g)(x). \end{aligned}$$

Using the square function estimate, we have

$$\begin{aligned} \|BHT \otimes \Pi \otimes \dots \Pi(f, g)\|_r &\lesssim \\ &\left\| \left(\sum_{k_1, \dots, k_{l+d}} \left| \sum_{k_{l+d+1}, \dots, k_n} Q_{k_{l+d+1}}^{l+d+1} \dots Q_{k_n}^n BHT(Q_{k_1}^{y_1} \dots Q_{k_l}^{y_l} P_{k_{l+1}}^{y_{l+1}} \dots P_{k_n}^{y_n} f, P_{k_1}^{y_1} \dots P_{k_l}^{y_l} Q_{k_{l+1}}^{y_{l+1}} \dots Q_{k_{l+d}}^{y_{l+d}} P_{k_{l+d+1}}^{y_{l+d+1}} \dots P_{k_n}^{y_n} g) \right|^2 \right)^{1/2} \right\|_r \\ &\lesssim \left\| \left(\sum_{k_1, \dots, k_{l+d}} \left| \sum_{k_{l+d+1}, \dots, k_n} |BHT(Q_{k_1}^{y_1} \dots Q_{k_l}^{y_l} P_{k_{l+1}}^{y_{l+1}} \dots P_{k_n}^{y_n} f, P_{k_1}^{y_1} \dots P_{k_l}^{y_l} Q_{k_{l+1}}^{y_{l+1}} \dots Q_{k_{l+d}}^{y_{l+d}} P_{k_{l+d+1}}^{y_{l+d+1}} \dots P_{k_n}^{y_n} g)|^2 \right| \right)^{1/2} \right\|_r \\ &\lesssim \|f\|_p \|g\|_q \end{aligned}$$

For the last part we used the following vector-valued estimates for the *BHT*:

$$\begin{aligned}
& L^p(\underbrace{l^\infty(\dots(l^\infty(l^2(\dots(l^2(l^2(\dots(l^2))\dots))\dots))}_{l} \times L^q(\underbrace{l^2(\dots(l^2(l^\infty(\dots(l^\infty(l^2(\dots(l^2))\dots))}_{d} \dots))_{n-l-d}) \\
& \mapsto L^s(\underbrace{l^2(\dots(l^2(l^2(\dots(l^2(l^1(\dots(l^1))\dots))}_{l} \dots))_{d} \dots))_{n-l-d})
\end{aligned}$$

together with boundedness of the maximal operator and square function.

The boundedness of the operator $\Pi \otimes \dots \otimes \Pi \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi$ is identical; it uses Fubini and the above vector-valued estimates for *BHT*. \square

6.3 Paraproducts on L^p spaces with mixed norms and new Leibniz rules

There will be two cases that we will thoroughly study, with different approaches:

- a) $\Pi \otimes \Pi : L_x^{p_1} L_y^{p_2} \times L_x^{q_1} L_y^{q_2} \rightarrow L_x^{s_1} L_y^{s_2}$, in the most general setting (i.e. no assumptions on the p_i, q_i, s_i .)
- b) $\Pi \otimes \dots \otimes \Pi : L_{x_1}^{p_1} \dots L_{x_n}^{p_n} \times L_{x_1}^{q_1} \dots L_{x_n}^{q_n} \rightarrow L_{x_1}^{s_1} \dots L_{x_n}^{s_n}$, for $1 < p_i, q_i, s_i < \infty$.

We recall the following results about square functions and maximal functions, which was presented in chapter 2:

Theorem 30. (c) Let $J_S \subseteq \{1, \dots, M\}$ and $J_M \subseteq \{1, \dots, M\}$ be two disjoint subsets of indices. We consider the operator $P_{J_S} Q_{J_M} f$ given by Littlewood-Paley projections in every component $j_s \in J_S$ and maximal operator in every component $j_m \in J_M$;

this is a bounded operator on $L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}$, provided $1 < p_1, \dots, p_M < \infty$:

$$\|\mathcal{S}^{J_S} \mathcal{M}^{J_M} f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}} \lesssim \|f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}}.$$

(d) Moreover, if there are only square functions appearing, we have an equivalence of norms:

$$\|\mathcal{S}^{J_S} f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}} \simeq \|f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} \dots L_{x_M}^{p_M}}.$$

Theorem 31. Let $1 < p_j, q_j \leq \infty$, $1 \leq s_j < \infty$, so that $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{s_j}$, $1 \leq j \leq 2$. Then

$$\|\Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2}} \lesssim \|f\|_{L_x^{p_1} L_y^{p_2}} \|g\|_{L_x^{q_1} L_y^{q_2}}.$$

Proof. First, note that

$$\|\Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Pi \otimes \Pi(f, g)(x, y)|^{s_1} dx \right)^{\frac{s_2}{s_1}} dy \right)^{1/s_2}.$$

There are two ways of regarding $\Pi \otimes \Pi$, in the sense that

$$\Pi_1 \otimes \Pi_2(f, g)(x, y) = \sum_l P_l^1 \Pi_2(Q_l^x f, P_l^x g)(y) = \sum_k P_k^2 \Pi_1(P_k^y f, Q_k^y g)(x).$$

If we choose so split the second paraproduct Π_2 , the range we can get is $1 < p_j, q_j < \infty$ and we illustrate the method below. Using the observations from section 6.2, we will regard the inner integral as a vector-valued tensor product. This will depend on the type of the second tensor paraproduct; we will present in detail the cases

- (i) $\Pi(f, g)(y) = \sum_k Q_k^2(P_k^2 f, P_k^2 g)(y)$ and
- (ii) $\Pi(f, g)(y) = \sum_k P_k^2(Q_k^2 f, Q_k^2 g)(y).$

In the case (i), $\Pi \otimes \Pi(f, g)(x, y) = \sum_k \Pi(P_k^y f, P_k^y g)(x)$, where $(P_k^y f)(x) = (\psi_k * f^x)(y)$.

$$\begin{aligned} \|\Pi \otimes \Pi(f, g)(\cdot, y)\|_{L_x^{s_1}} &= \left\| \sum_k \Pi(P_k^y f, P_k^y g) \right\|_{L_x^{s_1}} \lesssim \\ &\lesssim \left\| \left(\sum_k |P_k^y f|^2 \right)^{1/2} \right\|_{L_x^{p_1}} \cdot \left\| \left(\sum_k |P_k^y g|^2 \right)^{1/2} \right\|_{L_x^{q_1}}. \end{aligned}$$

Applying Hölder's inequality as well, we get the following estimate for $\|\Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2}}$:

$$\|\Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2}} \lesssim \left\| \left(\sum_k |P_k^y f(x)|^2 \right)^{1/2} \right\|_{L_x^{p_1}} \left\| \left(\sum_k |P_k^y g(x)|^2 \right)^{1/2} \right\|_{L_x^{q_1}} \left\| \cdot \right\|_{L_y^{s_2}}.$$

The operators on the RHS are vector-valued square functions, and their norms are bounded by $\|f\|_{L_x^{p_1} L_y^{p_2}}$ and $\|g\|_{L_x^{q_1} L_y^{q_2}}$, respectively.

Remark 19. In this case, we can have $\frac{1}{2} < s_1, s_2 \leq 1$ as well.

For the second case (ii), we could either use duality to reduce it to the previous example, or use part (d) of Theorem 30:

$$\begin{aligned} \|\Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2}} &= \left\| \sum_k \Pi \otimes P_k^2(f, g) \right\|_{L_x^{s_1} L_y^{s_2}} \lesssim \left\| \left(\sum_k |\Pi \otimes P_k^2(f, g)|^2 \right)^{1/2} \right\|_{L_x^{s_1} L_y^{s_2}} = \\ &= \left\| \left(\sum_k |\Pi(P_k^y f, Q_k^y g)(x)|^2 \right)^{1/2} \right\|_{L_x^{s_1} L_y^{s_2}} \lesssim \left\| \left(\sum_k |P_k^y f|^2 \right)^{1/2} \right\|_{L_x^{p_1}} \cdot \left\| \sup_k |Q_k^y g| \right\|_{L_x^{q_1}} \left\| \cdot \right\|_{L_y^{s_2}} \lesssim \\ &\lesssim \left\| \left(\sum_k |P_k^y f|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2}} \cdot \left\| \sup_k |Q_k^y f| \right\|_{L_x^{q_1} L_y^{q_2}}. \end{aligned}$$

Now we choose to split the first paraproduct.

(1) The most difficult case is when Π_1 and Π_2 are of the form

$$\begin{aligned} \Pi_1 &= \sum_l P_l(Q_l f \cdot P_l g) \quad \Pi_2 = \sum_k P_k(P_k f \cdot Q_k g). \quad \text{We have} \\ \Pi_1 \otimes \Pi_2(f, g)(x, y) &= \sum_l P_l^1 \Pi_2(Q_l^x f, P_l^x g)(y) = \sum_k P_k^2 \Pi_1(P_k^y f, Q_k^y g)(x). \end{aligned}$$

Using the first description, we have

$$\begin{aligned}
||| \Pi_1 \otimes \Pi_2 |||_{L_x^{s_1} L_y^{s_2}} &= ||| \sum_l P_l^1 \Pi_2(Q_l^x f, P_l^x g)(y) |||_{L_x^{s_1} L_y^{s_2}} \lesssim \\
&\lesssim ||| \left(\sum_l |\Pi_2(Q_l^x f, P_l^x g)(y)|^2 \right)^{1/2} |||_{L_x^{s_1} L_y^{s_2}} \lesssim \\
&\lesssim ||| \sup_l |Q_l^x f(y)| |||_{L_x^{p_1} L_y^{p_2}} \cdot ||| \left(\sum_l |P_l^x g(y)|^2 \right)^{1/2} |||_{L_x^{q_1} L_y^{q_2}} \lesssim \\
&\lesssim ||| f |||_{L_x^{p_1} L_y^{p_2}} \cdot ||| g |||_{L_x^{q_1} L_y^{q_2}}
\end{aligned}$$

We are using the following vector-valued estimates for paraproducts:

$$L_y^{p_2}(L_x^{p_1}(l^\infty)) \times L_y^{q_2}(L_x^{q_1}(l^2)) \rightarrow L_y^{s_2}(L_x^{s_1}(l^2)).$$

Constraints: $1 \leq s_1 < \infty, \frac{1}{2} < s_2 < \infty, 1 < p_i, q_i, 1 < q_1 < \infty$.

If $q_1 = \infty$ and $s_2 \geq 1$, we are interested in $L_x^{p_1} L_y^{p_2} \times L_x^\infty L_y^{q_2} \rightarrow L_x^{p_1} L_y^{s_2}$, and by duality it becomes

$$L_x^{p_1} L_y^{p_2} \times L_x^{p'_1} L_y^{s'_2} \rightarrow L_x^1 L_y^{q'_2}.$$

(2) Now we consider the case $\Pi_1(f, g) = \sum_l P_l(Q_l f \cdot P_l g)$, $\Pi_2 = \sum_k Q_k(P_k f \cdot P_k g)$.

$$\begin{aligned}
||| \Pi_1 \otimes \Pi_2 |||_{L_x^{s_1} L_y^{s_2}} &= ||| \sum_k Q_k^2 \Pi_1(P_k^y f, P_k^y g)(x) |||_{L_x^{s_1} L_y^{s_2}} \lesssim \\
&\lesssim ||| \sum_k |\Pi_1(P_k^y f, P_k^y g)(x)|^2 |||_{L_x^{s_1} L_y^{s_2}} \lesssim \\
&\lesssim ||| \left(\sum_k |P_k^y f(x)|^2 \right)^{1/2} |||_{L_x^{p_1}} \cdot ||| \left(\sum_k |P_k^y g(x)|^2 \right)^{1/2} |||_{L_x^{q_1} L_y^{s_2}} \lesssim \\
&\lesssim ||| \left(\sum_k |P_k^y f(x)|^2 \right)^{1/2} |||_{L_x^{p_1} L_y^{p_2}} \cdot ||| \left(\sum_k |P_k^y g(x)|^2 \right)^{1/2} |||_{L_x^{q_1} L_y^{q_2}}
\end{aligned}$$

Initially we cannot have an inner-most space be L^∞ (that is, $p_1 = \infty$ or $q_1 = \infty$), because the square functions are not bounded on L^∞ . In addition,

at most one of p_1 and q_1 can be ∞ . Say we have $q_1 = \infty$. By duality, the case

$$L_x^{p_1} L_y^{p_2} \times L_x^\infty L_y^{q_2} \rightarrow L_x^{p_1} L_y^{s_2} \quad \text{becomes} \quad L_x^{p_1} L_y^{p_2} \times L_x^{p'_1} L_y^{s'_2} \rightarrow L_x^1 L_y^{q'_2}.$$

□

Theorem 32. Let $1 < p_j, q_j < \infty$, $1 \leq s_j < \infty$, so that $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{s_j}$, $1 \leq j \leq M$. Then

$$\|\Pi \otimes \dots \otimes \Pi(f, g)\|_{L_{x_1}^{s_1} \dots L_{x_M}^{s_M}} \lesssim \|f\|_{L_{x_1}^{p_1} \dots L_{x_M}^{p_M}} \|g\|_{L_{x_1}^{q_1} \dots L_{x_M}^{q_M}}.$$

Proof. The proof is based on the iterated vector-valued estimates for paraproducts Π from section [4.3]. For simplicity, we illustrate the case $M = 3$, for a certain type of paraproduct, when $\Pi^2 = \sum_k Q_k^2(P_k^2, P_k^2)$, and $\Pi^3 = \sum_l P_l^3(Q_l^3, P_l^3)$. As before, we don't show the exterior Q_k^2 term because its Fourier transform is $\equiv 1$ on the frequency support of the input function. Then

$$\Pi \otimes \Pi \otimes \Pi(f, g)(x, y, z) = \sum_k \sum_l P_l^3 \Pi(P_k^y Q_l^z f, P_k^y P_l^z g)(x)$$

and using theorem 30, we have

$$\begin{aligned} \|\Pi \otimes \Pi \otimes \Pi(f, g)\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} &\simeq \left\| \left(\sum_l \left| \sum_k \Pi(P_k^y Q_l^z f, P_k^y P_l^z g)(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} \lesssim \\ &\lesssim \left\| \left\| \sup_l \left(\sum_k |P_k^y Q_l^z f|^2 \right)^{\frac{1}{2}} \right\|_{L_x^{p_1}} \cdot \left\| \left(\sum_l \sum_k |P_k^y P_l^z g|^2 \right)^{\frac{1}{2}} \right\|_{L_x^{q_1}} \right\|_{L_y^{s_2} L_z^{s_3}} \lesssim \\ &\lesssim \left\| \sup_l \left(\sum_k |P_k^y Q_l^z f|^2 \right)^{\frac{1}{2}} \right\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \cdot \left\| \left(\sum_l \sum_k |P_k^y P_l^z g|^2 \right)^{\frac{1}{2}} \right\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}} \lesssim \\ &\lesssim \|f\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \|g\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}}. \end{aligned}$$

□

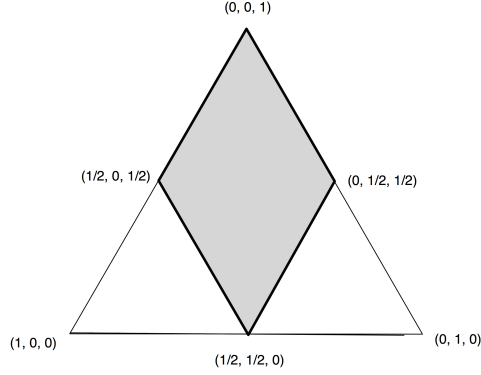


Figure 6.2: Range for $\Pi \otimes \dots \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi$ in L^p spaces with mixed norms

The same methods yield some partial results for mixed norm estimates for $BHT \otimes \Pi \dots \otimes \Pi$ and $\Pi \otimes \dots \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi$. Recall that the range for vector-valued BHT is conditioned very much by the triple (r_1, r_2, r) , so we cannot hope to obtain the full range for $\Pi \otimes \dots \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi$; not even the case when all the indices are in $(1, \infty)$ can be achieved. However, if $2 \leq p_j, q_j < \infty$, and $1 < s_j < \infty$, the question is fully answered.

Theorem 33. *For any $2 \leq p_j, q_j < \infty$, $1 < s_j < \infty$,*

$$\left\| \Pi \otimes \dots \otimes BHT \otimes \Pi \otimes \dots \otimes \Pi(f, g) \right\|_{L_{x_1}^{s_1} \dots L_{x_M}^{s_M}} \lesssim \|f\|_{L_{x_1}^{p_1} \dots L_{x_M}^{p_M}} \|g\|_{L_{x_1}^{q_1} \dots L_{x_M}^{q_M}}.$$

Proof. We illustrate the method of the proof in the case $M = 3$, and some particular choices of paraproducts. The other paraproducts case work in the similar way.

A) $BHT \otimes \Pi_2 \otimes \Pi_3$, where $\Pi_2 = \sum_k Q_k^2(P_k^2 \cdot P_k^2)$ and $\Pi_3 = \sum_l P_l^3(P_l^2 \cdot Q_l^3)$. Then

$$\begin{aligned} BHT \otimes \Pi_2 \otimes \Pi_3(f, g)(x, y, z) &= \sum_{k,l} P_l^3 Q_k^2 BHT(P_k^y P_l^z f, P_k^y Q_l^z g)(x) = \\ &= \sum_l P_l^3 \left(\sum_k Q_k^2 BHT(P_k^y P_l^z f, P_k^y Q_l^z g)(x) \right). \end{aligned}$$

$$\begin{aligned} \|BHT \otimes \Pi_2 \otimes \Pi_3\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} &\lesssim \left\| \left(\sum_l \left| \sum_k BHT(P_k^y P_l^z f, P_k^y Q_l^z g)(x) \right|^2 \right)^{1/2} \right\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} \lesssim \\ &\lesssim \left\| \left(\sum_{k,l} |P_k^y P_l^z f|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \cdot \left\| \sup_l \left(\sum_k |P_k^y Q_l^z g|^2 \right)^{1/2} \right\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}} \lesssim \\ &\lesssim \|f\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \cdot \|g\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}} \end{aligned}$$

Here we used

$$BHT : L_z^{p_3}(L_y^{p_2}(L_x^{p_1}(l^2(l^2)))) \times L_z^{q_3}(L_y^{q_2}(L_x^{q_1}(l^\infty(l^2)))) \rightarrow L_z^{s_3}(L_y^{s_2}(L_x^{s_1}(l^2(l^1)))).$$

and we know these always exist if $2 \leq p_1, p_2, q_1, q_2 \leq \infty$, and $1 \leq s_1, s_2 < \infty$.

But we combine these conditions with those for square functions, so we actually need $2 \leq p_1, p_2, q_1, q_2 < \infty$, and $1 < s_1, s_2 < \infty$.

B) $\Pi_1 \otimes BHT \otimes \Pi_3$, where $\Pi_1 = \sum_k Q_k^1(P_k^1 \cdot P_k^1)$ and $\Pi_3 = \sum_l P_l^3(P_l^2 \cdot Q_l^3)$. Then

$$\begin{aligned} \Pi_1 \otimes BHT \otimes \Pi_3(f, g)(x, y, z) &= \sum_{k,l} P_l^3 Q_k^1 BHT(P_k^x P_l^z f, P_k^x Q_l^z g)(x) = \\ &= \sum_l P_l^3 \left(\sum_k Q_k^1 BHT(P_k^x P_l^z f, P_k^x Q_l^z g)(y) \right) \end{aligned}$$

$$\begin{aligned} \|\Pi_1 \otimes BHT \otimes \Pi_3(f, g)\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} &\lesssim \left\| \left(\sum_l \left| \sum_k BHT(P_k^x P_l^z f, P_k^x Q_l^z g)(y) \right|^2 \right)^{1/2} \right\|_{L_x^{s_1} L_y^{s_2} L_z^{s_3}} \lesssim \\ &\lesssim \left\| \left(\sum_{k,l} |P_k^x P_l^z f|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2}} \cdot \left\| \sup_l \left(\sum_k |P_k^x Q_l^z g|^2 \right)^{1/2} \right\|_{L_x^{q_1} L_y^{q_2}} \lesssim \\ &\lesssim \left\| \left(\sum_{k,l} |P_k^x P_l^z f|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \cdot \left\| \sup_l \left(\sum_k |P_k^x Q_l^z g|^2 \right)^{1/2} \right\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}} \lesssim \\ &\lesssim \|f\|_{L_x^{p_1} L_y^{p_2} L_z^{p_3}} \cdot \|g\|_{L_x^{q_1} L_y^{q_2} L_z^{q_3}} \end{aligned}$$

Here we use $BHT : L_y^{p_2}(L_x^{p_1}(l^2(l^2))) \times L_y^{q_2}(L_x^{q_1}(l^\infty(l^2))) \rightarrow L_y^{s_2}(L_x^{s_1}(l^2(l^1)))$, and then take the norm in the z variable.

□

6.3.1 Leibniz Rules

Leibniz rules are very important in the theory of partial differential equations. We will be dealing with fractional derivatives, which are defined via the Fourier transform in the following way:

$$\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

The kind of estimates we can hope for are of the form

$$\|D^\alpha(f \cdot g)\|_s \lesssim \|D^\alpha(f)\|_p \cdot \|g\|_q + \|f\|_p \cdot \|D^\alpha(g)\|_q.$$

We know this estimates exist in $L^p(\mathbb{R}^d)$ spaces, due to multi-parameter paraproducts. We also know the following result from [23], which is due to Kenig, Ponce and Vega:

Theorem 34. *Let $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in [0, \alpha]$, with $\alpha = \alpha_1 + \alpha_2$. Let $1 < p_1, p_2, q_1, q_2, s_1, s_2 < \infty$ be such that $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{s_j}$. Then*

$$\|D_x^\alpha(f \cdot g) - f D_x^\alpha g - g D_x^\alpha f\|_{L_x^{s_1} L_T^{s_2}} \lesssim \|D^{\alpha_1} f\|_{L_x^{p_1} L_T^{p_2}} \|D^{\alpha_2} g\|_{L_x^{q_1} L_T^{q_2}}.$$

When dealing with derivative, Littlewood-Paley projections are very helpful because on the shell $|\xi_1| \sim 2^k$ taking an α -derivative D^α is equivalent to multi-

plying by $2^{\alpha k}$.

$$\begin{aligned} D_x^\alpha D_y^\beta (f \cdot g)(x, y) &= \sum_{k,l} [(f * \varphi_k \otimes \varphi_l) \cdot (g * \psi_l \otimes \psi_l)] * (D_x^\alpha \psi_k \otimes D_y^\beta \psi_l)(x, y) = \\ &= \sum_{k,l} [(f * \varphi_k \otimes \varphi_l) \cdot (g * \psi_k \otimes \psi_l)] * (2^{k\alpha} \tilde{\psi}_k \otimes 2^{l\beta} \tilde{\psi}_l)(x, y), \end{aligned}$$

where $\widehat{\tilde{\psi}}_k(\xi) = \frac{|\xi|^\alpha}{2^{k\alpha}} \widehat{\psi}_k(\xi)$ and $\widehat{\tilde{\psi}}_l(\eta) = \frac{|\eta|^\beta}{2^{l\beta}} \widehat{\psi}_l(\eta)$. The next step would be to move the derivatives inside, by moving the $2^{k\alpha}$ inside. However, we cannot move it onto a function of type φ and hope to obtain a derivative, because in that case we would have function $\widehat{\tilde{\varphi}}_k(\xi) = \frac{2^{k\alpha}}{|\xi|^\alpha} \widehat{\varphi}_k(\xi)$ appearing. Recall that $\widehat{\varphi}_k$ is supported on $|\xi| \leq 2^k$, so the origin could be contained inside its support. For this reason, whenever we are moving the $2^{k\alpha}$ inside, we want to attach it to a ψ -type function. Hence

$$\begin{aligned} D_x^\alpha D_y^\beta (f \cdot g)(x, y) &= \sum_{k,l} [f * (\varphi_k \otimes \varphi_l) \cdot g * (2^{k\alpha} \psi_k \otimes 2^{l\beta} \psi_l)] * (\tilde{\psi}_k \otimes \tilde{\psi}_l)(x, y) = \\ &= \sum_{k,l} [(f * \varphi_k \otimes \varphi_l) \cdot (D_x^\alpha D_y^\beta g * \tilde{\psi}_k \otimes \tilde{\psi}_l)] * (\tilde{\psi}_k \otimes \tilde{\psi}_l)(x, y). \end{aligned}$$

The expression $\tilde{\psi}_k := \frac{2^{k\alpha}}{|\xi|^\alpha} \widehat{\psi}_k(\xi)$ is okay because $\widehat{\psi}_k$ is supported on $|\xi| \sim 2^k$, which doesn't contain the origin. So we can write

$$D_x^\alpha D_y^\beta (f \cdot g) = \Pi \otimes \Pi(f, D_x^\alpha D_y^\beta g).$$

In order to prove the boundedness of the above expression in $L_x^{s_1} L_y^{s_2}$ by using the methods in 6.3, however, we need paraproducts that have the Fourier transform of the last ψ_k or φ_k identically equal to 1 on the Fourier support of the first part ($Q_k f \cdot P_k g$ or $P_k f \cdot P_k g$). This is important, because in this way we can regard the tensor product as a vector-valued paraproduct. We can't expect $\tilde{\psi}_k$ to have this property, so instead, we will use the Fourier series decomposition for $\widehat{\tilde{\psi}}_k$:

$$\widehat{\tilde{\psi}}_k(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{\frac{2\pi i n \xi}{2^k}}, \quad \text{where} \quad c_n = \frac{1}{2^k} \int_{\mathbb{R}} \widehat{\tilde{\psi}}_k(\xi) e^{-\frac{2\pi i n \xi}{2^k}} d\xi.$$

Because of the scaling invariance, the Fourier coefficients c_n don't depend on k ; moreover, the smoothness of the function $\widehat{\psi}$ implies a fast decay of these coefficients: $|c_n| \lesssim n^{-M}$. Now we have to deal with the operator

$$(F, G) \mapsto \sum_{n \in \mathbb{Z}} c_n \sum_{k, l} \left[F * (\varphi_k \otimes \varphi_l) \cdot G * (\tilde{\psi}_k \otimes \tilde{\psi}_l) \right] * \psi_{k, n} \otimes \tilde{\psi}_l(x, y).$$

Here $\psi_{k, n}(x) = \psi_k(x + 2^{-k}n)$. Now notice that the RHS above becomes

$$\begin{aligned} & \sum_n c_n \sum_{k, l} \left[F * (\varphi_k \otimes \varphi_l) \cdot G * (\tilde{\psi}_k \otimes \tilde{\psi}_l) \right] * \psi_k \otimes \tilde{\psi}_l(x, y) = \\ & := \sum_n c_n \sum_k P_k^1 \tilde{\Pi}(Q_{k, n}^x F, \tilde{P}_{k, n}^x G). \end{aligned}$$

The proof of the Leibniz rule follows from

1) (multiple) vector-valued estimates for the paraproduct

$$\tilde{\Pi}(f, g) = \sum_k \left[(f * \varphi_k) \cdot (g * \tilde{\psi}_k) \right] * \tilde{\psi}_k$$

2) the boundedness of the shifted maximal and square functions:

$$\| \sup_k |f * \varphi_{k, n}| \|_p \lesssim \log \langle n \rangle \|f\|_p, \quad \left\| \left(\sum_k |f * \tilde{\psi}_{k, n}|^2 \right)^{1/2} \right\|_p \lesssim \log \langle n \rangle \|f\|_p$$

Returning to Leibniz rule, we have for $s_1, s_2 \geq 1$

$$\begin{aligned} & \| \| \| D_x^\alpha D_y^\beta (f, g) \| \|_{L_x^{s_1} L_y^{s_2}} \leq \sum_n |c_n| \| \| \sum_k P_k^1 \tilde{\Pi}(Q_{k, n}^x F, \tilde{P}_{k, n}^x G) \| \|_{L_x^{s_1} L_y^{s_2}} \lesssim \\ & \lesssim \sum_n |c_n| \left\| \left(\sum_k |\tilde{\Pi}^{\beta_1, \beta_2}(Q_{k, n}^x F, \tilde{P}_{k, n}^x G)|^2 \right)^{1/2} \right\|_{L_x^{s_1} L_y^{s_2}} \lesssim \\ & \lesssim \sum_n |c_n| \sup_k \| Q_{k, n}^x F \|_{L_x^{p_1} L_y^{p_2}} \left\| \left(\sum_k |\tilde{P}_{k, n}^x G|^2 \right)^{1/2} \right\|_{L_x^{q_1} L_y^{q_2}} \lesssim \\ & \lesssim \|f\|_{L_x^{p_1} L_y^{p_2}} \| \| D_x^\alpha D_y^\beta g \| \|_{L_x^{q_1} L_y^{q_2}}. \end{aligned}$$

A more difficult case of the Leibniz rule is when one of the last components is a φ -type function:

$$\begin{aligned} D_x^\alpha D_y^\beta (f \cdot g)(x, y) &= \sum_{k,l} [(f * \psi_k \otimes \varphi_l) \cdot (g * \psi_l \otimes \psi_l)] * (D_x^\alpha \varphi_k \otimes D_y^\beta \psi_l)(x, y) = \\ &= \sum_{k,l} [(f * \psi_k \otimes \varphi_l) \cdot (g * \psi_k \otimes \psi_l)] * (2^{k\alpha} \tilde{\varphi}_k \otimes 2^{l\beta} \tilde{\psi}_l)(x, y). \end{aligned}$$

In this case too $\widehat{\tilde{\varphi}}_k = \frac{|\xi|^\alpha}{2^{k\alpha}} \hat{\varphi}_k(\xi)$, but $\tilde{\varphi}$ doesn't behave as nicely as $\tilde{\psi}_k$. For example, it decays much slower:

$$|\tilde{\varphi}(x)| \leq \frac{1}{(1 + |x|)^{1+\alpha}}.$$

The reason for this is that $|\xi|^\alpha$ is not smooth at the origin, and smoothness of \hat{f} corresponds to fast decay of f and vice versa.

We use the Fourier decomposition of $\widehat{\tilde{\varphi}}_k$:

$$\widehat{\tilde{\varphi}}_k(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{\frac{2\pi i n \xi}{2^k}} \cdot \hat{\varphi}_k(\xi), \quad \text{where} \quad c_n = \frac{1}{2^k} \int_{\mathbb{R}} \widehat{\tilde{\varphi}}_k(\xi) e^{-\frac{2\pi i n \xi}{2^k}} d\xi.$$

In this case we only have $|c_n| \leq \frac{1}{n^{1+\alpha}}$, but luckily this is enough for the coefficients to sum up, in the case where all the L^s spaces are Banach.

Following the same line of ideas, we could get

$$\begin{aligned} D_x^\alpha D_y^\beta (f \cdot g)(x, y) &= \sum_{k,l} [f * (2^{k\alpha} \psi_k \otimes \varphi_l) \cdot g * (\psi_k \otimes 2^{l\beta} \psi_l)] * (\tilde{\varphi}_k \otimes \tilde{\psi}_l)(x, y) = \\ &= \sum_{k,l} [(D_x^\alpha f * \tilde{\psi}_k \otimes \varphi_l) \cdot (D_y^\beta g * \psi_k \otimes \tilde{\psi}_l)] * (\tilde{\varphi}_k \otimes \tilde{\psi}_l)(x, y). \end{aligned}$$

We are interested in:

$$\begin{aligned} \sum_n c_n \sum_{k,l} [F * (\tilde{\psi}_{k,n} \otimes \varphi_l) \cdot G * (\psi_{k,n} \otimes \tilde{\psi}_l)] * \varphi_k \otimes \tilde{\psi}_l(x, y) = \\ = \sum_n c_n \sum_k \mathcal{Q}_k^1 \tilde{\Pi}(\tilde{P}_{k,n}^x F, P_{k,n}^x G)(y). \end{aligned}$$

This Leibniz rule can be deduced using vector-valued paraproducts:

$$\begin{aligned} \|D_x^\alpha D_y^\beta(f \cdot g)\|_{L_x^{s_1} L_y^{s_2}} &\lesssim \sum_n |c_n| \left\| \sum_k |\tilde{\Pi}(\tilde{P}_{k,n}^x F, P_{k,n}^x G)(y)| \right\|_{L_x^{s_1} L_y^{s_2}} \lesssim \\ &\lesssim \left\| \left(\sum_k |\tilde{P}_{n,k}^x|^2 \right)^{1/2} \right\|_{L_x^{p_1} L_y^{p_2}} \left\| \left(\sum_k |P_{n,k}^x|^2 \right)^{1/2} \right\|_{L_x^{q_1} L_y^{q_2}} \end{aligned}$$

We are left with the case of two φ -type functions being on the outer-most component:

$$\begin{aligned} D_x^\alpha D_y^\beta(f \cdot g)(x, y) &= \sum_{k,l} [(f * \psi_k \otimes \psi_l) \cdot (g * \psi_l \otimes \psi_l)] * (D_x^\alpha \varphi_k \otimes D_y^\beta \varphi_l)(x, y) = \\ &= \sum_{k,l} [(f * \psi_k \otimes \psi_l) \cdot (g * \psi_k \otimes \psi_l)] * (2^{k\alpha} \tilde{\varphi}_k \otimes 2^{l\beta} \tilde{\varphi}_l)(x, y). \end{aligned}$$

Both $\tilde{\varphi}_k$ and $\tilde{\varphi}_l$ have slower decay; if we use a Fourier decomposition for both $\widehat{\varphi}_k$ and $\widehat{\varphi}_l$, we obtain *shifted paraproduct*, for which we don't have any estimates.

Instead, we choose to split only one of them, say $\tilde{\varphi}_k$ to get in the end

$$\begin{aligned} \sum_n c_n \sum_{k,l} [F * (\tilde{\psi}_{k,n} \otimes \psi_l) \cdot G * (\psi_{k,n} \otimes \tilde{\psi}_l)] * \varphi_k \otimes \tilde{\varphi}_l(x, y) &= \\ = \sum_n c_n \sum_k Q_k^1 \tilde{\Pi}^\beta(\tilde{P}_{k,n}^x F, P_{k,n}^x G)(y). \end{aligned}$$

Note that $\tilde{\Pi}^\beta$ is a generalized paraproduct. The Leibniz rule will be a consequence of the vector-valued generalized paraproduct, with $r = 1$, which follows from 5.0.4.

We have the following theorem:

Theorem 35.

$$\begin{aligned} \|D_x^\alpha D_y^\beta(f \cdot g)\|_{L_x^{s_1} L_y^{s_2}} &\lesssim \|D_x^\alpha D_y^\beta f\|_{L_x^{p_1} L_y^{p_2}} \cdot \|g\|_{L_x^{q_1} L_y^{q_2}} + \|f\|_{L_x^{p_1} L_y^{p_2}} \cdot \|D_x^\alpha D_y^\beta g\|_{L_x^{q_1} L_y^{q_2}} + \\ &\quad + \|D_x^\alpha f\|_{L_x^{p_1} L_y^{p_2}} \cdot \|D_y^\beta g\|_{L_x^{q_1} L_y^{q_2}} + \|D_y^\beta f\|_{L_x^{p_1} L_y^{p_2}} \cdot \|D_x^\alpha g\|_{L_x^{q_1} L_y^{q_2}}, \end{aligned}$$

whenever $1 < p_j, q_j < \infty$, $1 \leq s_j < \infty$, and $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{s_j}$.

Remark 20. An equivalent statement is true for functions of n variables, as long as $1 < p_j, q_j < \infty$, $1 \leq s_j < \infty$, and $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{s_j}$. The idea of the proof is similar. We keep in mind that the outer-most expression will be

$$\tilde{\psi}_{k_1} \otimes \tilde{\psi}_{k_2} \otimes \dots \tilde{\psi}_{k_n} \quad \text{or} \quad \tilde{\psi}_{k_1} \otimes \tilde{\varphi}_{k_2} \otimes \dots \tilde{\varphi}_{k_n},$$

a combination of $\tilde{\psi}_{k_i}$ and $\tilde{\varphi}_{k_j}$ functions. As long as we have at least one $\tilde{\psi}_{k_i}$ function, we are using Fourier decomposition for all the other ones, and we also split the paraproducts in the x_j variable, with $j \neq i$. In the end we get a vector-valued paraproduct in many variables, where the vector space is going to be a combination of L^r and l^s spaces.

If the outer-most expression involves only $\tilde{\varphi}_{k_j}$ functions, we still split all of them but the inner-most function. This will yield a vector-valued paraproduct, where the vector space is $l^1(l^1(\dots l^1))$. But for this case we always have estimates:

$$\begin{aligned} & \sum_{k_1, \dots, k_n} [F * (\psi_{k_1} \otimes \dots \otimes \psi_{k_n}) \cdot G * (\psi_{k_1} \otimes \dots \otimes \psi_{k_n})] * (\tilde{\varphi}_{k_1} \otimes \dots \otimes \tilde{\varphi}_{k_n})(x_1, \dots, x_n) = \\ & = \sum_{m_2, \dots, m_n} c_{m_2} \dots c_{m_n} \sum_{k_2, \dots, k_n} Q_{k_2}^2 \dots Q_{k_n}^n \tilde{\Pi}^{\alpha_1} (P_{k_2}^{x_2} \dots P_{k_n}^{x_n} F, P_{k_2}^{x_2} \dots P_{k_n}^{x_n} G)(x_1). \end{aligned}$$

We recall that the c_{m_j} are Fourier coefficients, and $|c_{m_j}| \lesssim (1 + |m_j|)^{-1-\alpha_j}$, but we are in the Banach case, and the series will converge without a problem.

CHAPTER 7

ALTERNATIVE PROOFS

7.1 Square Functions

In this chapter, \mathcal{J} is a collection of intervals, $\{\psi_I\}_I$ is a collection of wave packets associates with \mathcal{J} , and $\{h_I\}_I$ are the Haar functions associated with them.

We will be studying the following square functions:

$$\begin{aligned}
 (1) \quad \mathcal{S}_1 f(x) &= \left(\sum_{I \in \mathcal{J}} |\langle f, \psi_I \rangle|^2 |\psi_I(x)|^2 \right)^{1/2} \\
 (2) \quad \mathcal{S}_2 f(x) &= \left(\sum_{I \in \mathcal{J}} |\langle f, \psi_I \rangle|^2 |h_I(x)|^2 \right)^{1/2} = \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} \\
 (3) \quad \mathcal{S}_3 f(x) &= \left(\sum_{I \in \mathcal{J}} |\langle f, h_I \rangle|^2 |h_I(x)|^2 \right)^{1/2} = \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} \\
 (4) \quad \mathcal{S}_4 f(x) &= \left(\sum_{I \in \mathcal{J}} |\langle f, h_I \rangle|^2 |\psi_I(x)|^2 \right)^{1/2}
 \end{aligned}$$

Because of its decay away from I , $\psi_I(\cdot)$ can be estimated by

$$|\psi_I(x)|^2 \lesssim \sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{100}} \mathbf{1}_{I_n}(x). \quad (7.1)$$

For any interval I , $I_n := I + n|I|$ denotes its translate n units to the left or right. We will also consider $h_{I_n}(\cdot)$ to be the Haar function associated with the dyadic interval I_n . Because of 7.1, we will be interested in studying the following shifted square functions:

$$(5) \quad \mathcal{S}_5 f(x) = \left(\sum_{I \in \mathcal{J}} |\langle f, h_I \rangle|^2 |h_{I_n}(x)|^2 \right)^{1/2} = \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_{I_n} \right)^{1/2}$$

$$(6) \quad \mathcal{S}_6 f(x) = \left(\sum_{I \in \mathcal{J}} |\langle f, \psi_I \rangle|^2 |h_{I_n}(x)|^2 \right)^{1/2} = \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \cdot \mathbf{1}_{I_n} \right)^{1/2}$$

The norm of the shifted square functions has an extra factor of $\log \langle n \rangle$; in the case of the regular square functions, the $\log \langle n \rangle$ will be cancelled by the fast decay of the ψ_I s, so the operatorial norm will not depend on n at all.

Associated to these square functions, we have the following singular operators:

$$\begin{aligned} 1) \quad T_1 f(x) &= \sum_I r_I(t) \langle f, \psi_I \rangle \psi_I(x) \\ 2) \quad T_2 f(x) &= \sum_I r_I(t) \langle f, \psi_I \rangle h_I(x) \\ 3) \quad T_3 f(x) &= \sum_I r_I(t) \langle f, h_I \rangle h_I(x) \\ 4) \quad T_4 f(x) &= \sum_I r_I(t) \langle f, h_I \rangle \psi_I(x) \\ 5) \quad T_5 f(x) &= \sum_I r_I(t) \langle f, h_I \rangle h_{I_n}(x) \\ 6) \quad T_6 f(x) &= \sum_I r_I(t) \langle f, \psi_I \rangle h_{I_n}(x) \end{aligned}$$

The connection is made through Khintchine's inequality:

$$\left\| \left(\sum_{I \in \mathcal{J}} |a_I \phi_I(x)|^2 \right)^{1/2} \right\|_p \sim \left(\int_0^1 \int_{\mathbb{R}} \left| \sum_{I \in \mathcal{J}} r_I(t) a_I \phi_I(x) \right|^p dx dt \right)^{1/p}.$$

Then note that $T_1^* = T_1, T_2^* = T_4, T_3^* = T_3$. The dual of T_5^* is a similar operator; we can consider the collection of intervals to be shifted.

7.1.1 Boundedness of \mathcal{S}_3

Definition 27. If I_0 is a fixed dyadic interval and \mathcal{J} is a collection of dyadic intervals, all contained in I_0 , we denote $\mathcal{J}^+(I_0)$ the set of all dyadic intervals $\tilde{I} \subseteq I_0$ with the property

that there exists an $I \in \mathcal{J}$ so that $I \subseteq \tilde{I} \subseteq I_0$. Then

$$\text{size}_{I_0} g := \sup_{I' \in \mathcal{J}^+(I_0)} \frac{1}{|3I'|} \int_{\mathbb{R}} |g(x)| \cdot \mathbf{1}_{3I'} dx.$$

Lemma 11. Let I_0 be a fixed dyadic interval as before; consider the localized square function

$$\mathcal{S}_3 f(x) = \left(\sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} |\langle f, h_I \rangle|^2 |h_I(x)|^2 \right)^{1/2} = \left(\sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2}.$$

For a Schwartz function $g \in \mathcal{S}$, we have

$$\left| \int_{\mathbb{R}} \left(\sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} g(x) dx \right| \lesssim \text{size}_{I_0} g \cdot \frac{\|f \cdot \mathbf{1}_{I_0}\|_2}{|I_0|^{1/2}} \cdot |I_0|. \quad (7.2)$$

Proof. Let \mathcal{J} be the collection of maximal dyadic intervals with the property that no $I \in \mathcal{J}$ is contained inside $3J$. This is a partition of \mathbb{R} , and the integral in the lemma can be expressed as

$$\sum_{J \in \mathcal{J}} \int_J \left(\sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} g(x) dx.$$

We denote $\mathcal{I}_{\text{shadow}} := \cup_{I \in \mathcal{J}} I \subseteq I_0$ the union of the intervals $I \in \mathcal{J}$. Then the intervals $J \in \mathcal{J}$ appearing in the expression above, they all intersect $\mathcal{I}_{\text{shadow}}$, and if $I \cap J \neq \emptyset$, we have $J \subsetneq I$ and $|J| < |I|$. This is an easy consequence of the structure of dyadic mesh. Then 7.2 becomes

$$\begin{aligned} \left| \sum_{J \in \mathcal{J}} \int_J \left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} g(x) dx \right| &\leq \\ &\leq \sum_{J \in \mathcal{J}} \int_J \left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} |g(x)| dx \end{aligned}$$

Here we need to make two remarks:

1) since $J \in \mathcal{J}$ was maximal with the property that no $I \subseteq 3J$, then it must be the case that there exists some $I(J) \in \mathcal{J}$ with the property that $I(J) \subset 3\tilde{J}$, where \tilde{J} denote the dyadic parent of J . So $I(J)$ is contained in one of the intervals $\tilde{J}, \tilde{\tilde{J}}$, or some translate(one unit to the right or left) of these. In any case, we can conclude that there exists a dyadic interval I' containing $I(J)$, so that $J \subset 3I'$, and so that $|J| < |I'| \leq 4|J|$.

2) The second observation is that, given a J with the above properties, any I appearing in the square function will contain J , and for that reason the expression

$$\left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2}$$

is constant on J . Moreover,

$$\left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} = \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} \cdot \mathbf{1}_J(x).$$

Hence we can say that

$$\begin{aligned} & \int_J \left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} |g(x)| dx \lesssim \\ & \lesssim \frac{1}{|3I'|} \int_{\mathbb{R}} \mathbf{1}_{3I'}(x) |g(x)| dx \cdot |J| \cdot \sup_{x \in J} \left(\sum_{\substack{I \in \mathcal{J} \\ |I| > |J|}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} \lesssim \text{size}_{I_0} g \int_J \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2}. \end{aligned}$$

Summing over J , and given that all $J \subseteq I_0$, we conclude

$$\left| \int_{\mathbb{R}} \left(\sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} g(x) dx \right| \lesssim \text{size}_{I_0} g \cdot \frac{1}{|I_0|^{1/2}} \cdot \left\| \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} \right\|_2 \cdot |I_0|.$$

Then the conclusion follows from the L^2 boundedness of the square func-

tions:

$$\left\| \left(\sum_I |a_I h_I(x)|^2 \right)^{1/2} \right\|_2 = \left(\sum_I |a_I|^2 \right)^{1/2} = \left\| \sum_I a_I h_I(x) \right\|_2.$$

This only uses basic properties about the Haar system, more exactly that it is an orthonormal system. \square

Proposition 21. S_3 is a bounded operator from L^p to L^p provided $p \geq 2$.

Proof. Using restricted type interpolation, let $|f| \leq \mathbf{1}_F$ and $|g| \leq \mathbf{1}_G$. We want

$$\left| \int_{\mathbb{R}} \left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I \right)^{1/2} g(x) dx \right| \lesssim |F|^{1/p} |G|^{1/p'}.$$

The proof is based on a stopping time argument, which has to be performed with care.

For f , we denote \mathcal{J}_{n_1} the collection of maximal dyadic intervals having the property that it contains some $I \in \mathcal{J}$ which is not contained in any other interval chosen previously, and so that

$$2^{-n_1-1} \leq \frac{1}{|I_0|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{I_0}(x) dx \leq 2^{-n_1} \leq 1. \quad (7.3)$$

Also, it is important to make sure that for any other interval $\tilde{I} \subset I_0$ which contains some $I \in \mathcal{J}$, we have

$$\frac{1}{|\tilde{I}|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{\tilde{I}}(x) dx \leq 2^{-n_1}.$$

We start with the largest possible value $2^{-n_1} \leq 1$, and continue until we exhaust all the intervals in \mathcal{J} . In the end, for a fixed value n_1 , all the intervals in \mathcal{J}_{n_1} are mutually disjoint, and moreover

$$\sum_{I \in \mathcal{J}_{n_1}} |I| \leq 2^{n_1} |F|.$$

For g , the stopping time is done with respect to

$$\text{size}_{I_0} g := \sup_{\substack{I' \subseteq I_0 \\ I' \in \mathcal{J}^+(I_0)}} \frac{1}{|3I'|} \int_{\mathbb{R}} \mathbf{1}_{3I'}(x) |g(x)| dx \lesssim 1.$$

We start with the largest possible value $2^{-\tilde{n}_2}$ of $\frac{1}{|3I|} \int_{\mathbb{R}} \mathbf{1}_{3I}(x) |g(x)| dx$, where $I \in \mathcal{J}$; say supremum is attained on I_0 , which is also chosen so that it is maximal with respect to inclusion. Then we look for dyadic intervals \tilde{I} containing I_0 which also have the property that

$$2^{-\tilde{n}_2-1} \leq \frac{1}{|3\tilde{I}|} \int_{\mathbb{R}} \mathbf{1}_{3\tilde{I}}(x) |g(x)| dx \leq 2^{-\tilde{n}_2}.$$

Pick \tilde{I}_0 , which has the above property and also is maximal. Note that for any $I' \subset \tilde{I}_0$, $\text{size}_{I'} g \leq \text{size}_{\tilde{I}_0} g \sim 2^{-\tilde{n}_2}$. Then we continue the procedure, making sure that the “top” intervals \tilde{I}_0 are disjoint. Then $\mathcal{J}_{\tilde{n}_2}$ will be the collection of these intervals \tilde{I}_0 .

The boundedness of the maximal operator will imply again that

$$\sum_{\tilde{I}_0} |\tilde{I}_0| \lesssim 2^{\tilde{n}_2} |G|.$$

Then we do the same with all $2^{-n_2} \leq 2^{-\tilde{n}_2}$, obtaining collections \mathcal{J}_{n_2} .

Returning to the square function, we first dualize the l^2 norm:

$$\left(\sum_{I \in \mathcal{J}} \frac{|\langle f, h_I \rangle|^2}{|I|} \cdot \mathbf{1}_I(x) \right)^{1/2} = \sum_{I \in \mathcal{J}} \epsilon_I(x) \langle f, h_I \rangle h_I(x).$$

Here the $\epsilon_I(x)$ functions have the property that $\left(\sum_I |\epsilon_I(x)|^2 \right)^{1/2} = 1$ for a.e. x . The dualization will allow us to change the order of summation, or break down the operator in multiple sums.

$$\int_{\mathbb{R}} \sum_{I \in \mathcal{J}} \epsilon_I(x) \langle f, h_I \rangle h_I(x) g(x) dx = \sum_{n_1, n_2} \sum_{I_0 \in \mathcal{J}^{n_2, n_2}} \int_{\mathbb{R}} \sum_{\substack{I \in \mathcal{J} \\ I \subseteq I_0}} \epsilon_I(x) \langle f, h_I \rangle h_I(x) g(x) dx.$$

The local estimate in Lemma 11, and the stopping times will imply

$$\begin{aligned} \left| \int_{\mathbb{R}} \sum_{I \in \mathcal{J}} \epsilon_I(x) \langle f, h_I \rangle h_I(x) g(x) dx \right| &\lesssim \sum_{n_1, n_2} \sum_{I_0 \in \mathcal{J}^{n_2, n_2}} 2^{-\frac{n_1}{2}} 2^{-n_2} |I_0| \lesssim \\ &\lesssim \sum_{n_1, n_2} 2^{-\frac{n_1}{2}} 2^{-n_2} (2^{n_1} |F|)^{\theta_1} (2^{n_2} |G|)^{\theta_2} \lesssim \sum_{n_1, n_2} 2^{-n_1(\frac{1}{2} - \theta_1)} 2^{-n_2(1 - \theta_2)} |F|^{\theta_1} |G|^{\theta_2}. \end{aligned}$$

Here $\theta_1 + \theta_2 = 1$, and we can actually choose them so that $\theta_1 \sim \frac{1}{p}$ and $\theta_2 \sim \frac{1}{p'}$.

There is however a constraint: we need $\frac{1}{2} > \frac{1}{p'}$, or equivalently $2 < p$.

The case $p = 2$ can be proved directly; therefore we obtain the range $p \geq 2$. □

Alternatively, we are going to present a proof for the boundedness of T_3 on L^p , for $p \geq 2$. Then the fact that T_3 is self-adjoint will imply that T_3 is bounded on L^p , for $1 < p < \infty$, and this implies the boundedness of S_3 within the same range. Before, the methods of the proof would only apply for $p \geq 2$.

Proposition 22. *T_3 is a bounded operator from L^p to L^p for any $1 < p < \infty$.*

Proof. The proof is almost identical to that in Proposition 21; but we can take advantage of the linearity of T_3 and deal more easily with its adjoint.

Recall $T(f)(x) = \sum_I r_I(t) \langle f, h_I \rangle h_I(x)$. We need to localize this operator; we will show

$$\left| \int_{\mathbb{R}} \sum_{I \subseteq I_0} r_I(t) \langle f, h_I \rangle h_I(x) g(x) dx \right| \lesssim \text{size}_{I_0} g \cdot \frac{\|f \cdot \mathbf{1}_{I_0}\|_2}{|I_0|^{1/2}} \cdot |I_0|. \quad (7.4)$$

To get this estimate we define the collection \mathcal{J} in the same way, and the only thing that need to be proved is that

$$\left(\int_{\mathbb{R}} \left| \sum_{I \subseteq I_0} r_I(t) \langle f, h_I \rangle h_I(x) \right|^2 dx \right)^{1/2} \lesssim \|f \cdot \mathbf{1}_{I_0}\|_2.$$

This can be showed without difficulty, using only the orthogonality of the $\{h_I\}$:

$$\begin{aligned} \left\| \sum_{I \subseteq I_0} r_I(t) \langle f, h_I \rangle h_I(x) \right\|_2^2 &= \left\langle \sum_{I \subseteq I_0} r_I(t) \langle f, h_I \rangle h_I, \sum_{I' \subseteq I_0} r_{I'}(t) \langle f, h_{I'} \rangle h_{I'}(x) \right\rangle = \\ &= \sum_{I \subseteq I_0} |\langle f, h_I \rangle|^2 = \sum_{I \subseteq I_0} \langle f, h_I \rangle \overline{\langle f, h_I \rangle} = \langle f, \sum_I \langle f, h_I \rangle h_I \rangle \leq \|f\|_2 \left\| \sum_I \langle f, h_I \rangle h_I \right\|_2 \leq \\ &\leq \|f\|_2 \left(\sum_I |\langle f, h_I \rangle|^2 \right)^{1/2}. \end{aligned}$$

These direct computations yield $\left(\sum_I |\langle f, h_I \rangle|^2 \right)^{1/2} \leq \|f\|_2$, and hence also the desired conclusion. Then the stopping time is identical, and so is the rest of the argument. \square

7.1.2 Boundedness of T_5

We will be looking at a slightly different operator, which we are also calling T_5 :

$$\sum_{I \in \mathcal{J}} r_I(t) \langle f, h_{I_n} \rangle h_I(x).$$

Proposition 23. *For any $1 < p < \infty$,*

$$\|T_5(f)\|_p \lesssim (\log \langle n \rangle)^2 \|f\|_p.$$

Proof. We start with the localization process: assume all the intervals $I \in \mathcal{J}$ are contained inside some fixed interval I_0 .

$$\int_{\mathbb{R}} \sum_{I \subseteq I_0} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx = \sum_{J \in \mathcal{J}} \int_J \sum_{I \subseteq I_0} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx$$

The collection \mathcal{J} is defined as before; and hence

$$|\sum_{I_0 \supset I \supset J} r_I(t) \langle f, h_{I_n} \rangle h_I(x)|$$

is constant on J . Eventually we obtain

$$|\int_{\mathbb{R}} \sum_{I \subset I_0} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx| \lesssim \text{size}_{I_0} g \cdot \|\sum_{I \subset I_0} r_I(t) \langle f, h_{I_n} \rangle h_I(x)\|_2 \cdot |I_0|^{1/2}.$$

The expression $\|\sum_{I \subset I_0} r_I(t) \langle f, h_{I_n} \rangle h_I(x)\|_2$ is bounded by $\|f \cdot \mathbf{1}_{I_{\text{shadow}}(\mathcal{J}(I_0))}\|_2$, where

$$I_{\text{shadow}(\mathcal{J}(I_0))} := \bigcup_{\substack{I \subset I_0 \\ I \in \mathcal{J}}} I_n \subset \bigcup_{l=0}^{\log \langle n \rangle} I_0^m.$$

The stopping time for the function f is going to be slightly convoluted, because of the shifted intervals; while the intervals $I \in \mathcal{J}(I_0)$ are all contained inside I_0 , their translates “spread out”. However, there are only $\log \langle n \rangle$ possible translates of I_0 where I_n can be. We denote these by I_0^m , where $I_0^m = I_{0,2^m}$, and $0 \leq m \leq \log \langle n \rangle$. Instead of estimating

$$\frac{1}{|I_0|} \|f \cdot \mathbf{1}_{\text{shadow}(\mathcal{J}(I_0))}\|_2^2,$$

we will be looking at the simpler quantity $\frac{|F \cap I_0^m|}{|I_0^m|} \leq 1$. The reason for this reduction is that $|f| \leq \mathbf{1}_F$, as we can assume.

To start the stopping time, let $\mathcal{J}_{\text{stock}} := \mathcal{J}$, and $2^{-\tilde{n}_1} := \sup_{I \in \mathcal{J}_{\text{stock}}} \frac{1}{|I_n|} \int_{\mathbb{R}} \mathbf{1}_F \cdot \mathbf{1}_{I_n} dx$. Because \mathcal{J} is a finite collection, \sup is attained at some interval \tilde{I} . Then \tilde{I} is contained in some interval I_1 , and $\tilde{I}_n \subseteq I_1^m := I_{1,2^m}$ is contained in one of the $\log \langle n \rangle$ translates of I_1 . However, we want to find the maximal interval I_1 containing \tilde{I} , and so that

$$2^{-\tilde{n}_1-1} \leq \frac{1}{|I_1|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{\text{shadow}(\mathcal{J}_{\text{stock}}(I_1), m)} dx \leq 2^{-\tilde{n}_1}.$$

Such an interval certainly exists because a candidate is \tilde{I} itself. Then we add I_1 to the collection $\mathcal{J}_{\tilde{n}_1, m}$ (which is initially empty), and we let

$$\mathcal{J}_{\tilde{n}_1, m}(I_1) := \{I \in \mathcal{J}_{Stock} : I \subseteq I_1, \text{ and } I_n \subseteq I_1^m\}.$$

Then set $\mathcal{J}_{Stock} := \mathcal{J}_{Stock} \setminus \mathcal{J}_{\tilde{n}_1, m}(I_1)$, and restart the procedure until all the intervals are exhausted.

Now it is important to obtain a comparison result; let $I_1 \in \mathcal{J}_{n_1, m}$, and $I_0 \subseteq I_1$ a subinterval. Because $I_1 \in \mathcal{J}_{n_1, m}$, we know that

$$2^{-n_1-1} \leq \frac{1}{|I_1|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{shadow(\mathcal{J}_{n_1, m}(I_1), m)} dx \leq 2^{-n_1},$$

but what can we say about

$$\frac{1}{|I_0|} \int_{\mathbb{R}} \mathbf{1}_F(x) \mathbf{1}_{shadow(\mathcal{J}(I_0) \cap \mathcal{J}_{n_1, m}(I_1))} dx?$$

The last expression looks rather complicated, but it emerges from

$$\frac{1}{|I_0|} \sum_{\substack{I \in \mathcal{J}_{n_1, m} \\ I \subseteq I_0 \subseteq I_1}} |\langle f, h_{I_n} \rangle|^2,$$

which appears naturally in the localization process.

We also note the subtle difference between $\mathcal{J}_{n_1, m}(I_1)$ and $\mathcal{J}(I_1)$. First, the n_1 denotes the fact that $I \in \mathcal{J}_{n_1, m}(I_1)$ was selected during the stopping time corresponding to the 2^{-n_1} average; there might be some other subintervals of I_1 that have been selected at a previous step. For example, if $\tilde{I}_1 \subsetneq I_1$ was selected in $\mathcal{J}_{n_1-1, l}$, then there will be intervals $I \in \mathcal{J}_{n_1-1, l}(\tilde{I}_1)$ which don't appear in $\mathcal{J}_{n_1, m}(I_1)$.

We look at $shadow(\mathcal{J}(I_0) \cap \mathcal{J}_{n_1, m}(I_1))$. If $I \in \mathcal{J}(I_0) \cap \mathcal{J}_{n_1, m}(I_1)$, then $I \subset I_0 \subset I_1$, and $I_n \in I_1^m$. Also, there are $\log \langle n \rangle$ translates of I_0 where I_n could possibly lie. So we can say

$$shadow(\mathcal{J}(I_0) \cap \mathcal{J}_{n_1, m}(I_1)) := \bigcup_{l=0}^{\log \langle n \rangle} I_0^l \cap shadow(\mathcal{J}_{n_1, m}).$$

For every $0 \leq l \leq \log\langle n \rangle$, we can find a decomposition of $I_0^l \cap \text{shadow}(\mathcal{J}_{n_1,m}) := \bigcup_{I \in \mathcal{A}} I_n$, made up of maximal (hence disjoint) intervals. Then

$$|F \cap I_0^l \cap \text{shadow}(\mathcal{J}_{n_1,m})| = \sum_{I \in \mathcal{A}} |F \cap I_n| \leq \sum_{I \in \mathcal{A}} I \leq 2^{-n_1} |I_0^l \cap \text{shadow}(\mathcal{J}_{n_1,m})|.$$

Here it is essential that an $I \in \mathcal{J}_{n_1,m}$ has not been selected before, and so $\frac{|F \cap I_n|}{|I_n|} \leq 2^{-n_1}$. This further implies that $|F \cap \text{shadow}(\mathcal{J}(I_0) \cap \mathcal{J}_{n_1,m}(I_1))| \leq 2^{-n_1} \log\langle n \rangle |I_0|$, or equivalently

$$\frac{1}{|I_0|} \int_{\mathbb{R}} \mathbf{1}_F(x) \mathbf{1}_{\text{shadow}(\mathcal{J}(I_0) \cap \mathcal{J}_{n_1,m}(I_1))} dx \leq 2^{-n_1}.$$

After creating the proper partitions of \mathcal{J} , we are ready to proceed with the proof.

One partition is

$$\mathcal{J} := \bigcup_{n_2 \geq 0} \{I \in \mathcal{J}_{n_2}(I_2), \text{ where size }_{I_2} \mathbf{1}_G \sim 2^{-n_2}\}.$$

The other partition is given by

$$\mathcal{J} := \bigcup_{m=0}^{\log\langle n \rangle} \bigcup_{n_1 \geq 0} \{I \in \mathcal{J}_{n_1,m}(I_1), \text{ where } I_1 \in \mathcal{J}_{n_1,m}\}.$$

$$\begin{aligned} & \left| \int_{\mathbb{R}} \sum_{I \in \mathcal{J}} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx \right| \leq \\ & \leq \sum_{m=0}^{\log\langle n \rangle} \sum_{n_1, n_2} \sum_{\substack{I_0 = I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1,m} \\ I_2 \in \mathcal{J}_{n_2}}} \left| \sum_{\substack{I \subset I_0 \\ I \in \mathcal{J}_{n_1,m}(I_1) \\ I \in \mathcal{J}_{n_2}(I_2)}} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx \right| \end{aligned}$$

Even though we can easily have control of the operator T_5 over the intervals $I_1 \in \mathcal{J}_{n_1,m}$, we had to make sure we can find good estimates when restrict it to the subinterval $I_1 \cap I_2$. This is the reason for the complicated stopping time. Using

the localization result, an upper bound for the operator is

$$\begin{aligned}
& \sum_{m=0}^{\log\langle n \rangle} \sum_{n_1, n_2} \sum_{\substack{I_0=I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1, m} \\ I_2 \in \mathcal{J}_{n_2}}} \text{size}_{I_0} g \cdot \left(\frac{1}{|I_0|} \int_{\mathbb{R}} \mathbf{1}_F(x) \mathbf{1}_{\text{shadow}(\mathcal{J}_{n_1, m}(I_1) \cap \mathcal{J}(I_0))}(x) dx \right)^{1/2} \cdot |I_0| \lesssim \\
& \lesssim \sum_{m=0}^{\log\langle n \rangle} \sum_{n_1, n_2} \sum_{\substack{I_0=I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1, m} \\ I_2 \in \mathcal{J}_{n_2}}} 2^{-n_2} (2^{-n_1} \log\langle n \rangle)^{1/2} |I_0|.
\end{aligned}$$

Now we estimate $\sum_{n_1, n_2} \sum_{\substack{I_0=I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1, m} \\ I_2 \in \mathcal{J}_{n_2}}} |I_0|$ in two different ways:

1)

$$\sum_{\substack{I_0=I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1, m} \\ I_2 \in \mathcal{J}_{n_2}}} |I_0| \leq \sum_{I_1 \in \mathcal{J}_{n_2}} |I_2| \lesssim 2^{n_2} |G| \quad (7.5)$$

2)

$$\sum_{\substack{I_0=I_1 \cap I_2 \\ I_1 \in \mathcal{J}_{n_1, m} \\ I_2 \in \mathcal{J}_{n_2}}} |I_0| \leq \sum_{I_1 \in \mathcal{J}_{n_1, m}} |I_1| = \left\| \sum_{I_1 \in \mathcal{J}_{n_1, m}} \mathbf{1}_{I_1} \right\|_{1, \infty} \lesssim 2^{n_1} \left\| \sum_{I_1 \in \mathcal{J}_{n_1, m}} \mathbf{1}_{I_1^m} \mathcal{M}_n \mathbf{1}_F \right\|_{1, \infty} \lesssim 2^{n_1} \log\langle n \rangle |F|. \quad (7.6)$$

For the last part, we used the following:

– the intervals $I_1 \in \mathcal{J}_{n_1, m}$ are disjoint, and for this reason $\sum_{I_1 \in \mathcal{J}_{n_1, m}} |I_1| =$

$$\left\| \sum_{I_1 \in \mathcal{J}_{n_1, m}} \mathbf{1}_{I_1} \right\|_{1, \infty}.$$

– for each $I_1 \in \mathcal{J}_{n_1, m}$,

$$2^{-n_1-1} \leq \frac{1}{|I_1|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{\text{shadow}(\mathcal{J}_{n_1, m}(I_1), m)} dx \leq \frac{1}{|I_1|} \int_{\mathbb{R}} \mathbf{1}_F(x) \cdot \mathbf{1}_{I_1^m} dx.$$

This means that $\mathbf{1}_{I_1} \leq 2^{n_1} \mathcal{M}_m(\mathbf{1}_F) \cdot \mathbf{1}_{I_1}$, and the boundedness of the shifted maximal operator implies

$$\sum_{I_1 \in \mathcal{J}_{n_1, m}} |I_1| \lesssim 2^{n_1} \log\langle m \rangle |F|.$$

All of the above imply

$$\begin{aligned} \left| \int_{\mathbb{R}} \sum_{I \in \mathcal{J}} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx \right| &\lesssim \sum_{m=0}^{\log \langle n \rangle} \sum_{n_1, n_2} 2^{-n_2} (2^{-n_1} \log \langle n \rangle)^{1/2} (2^{n_1} \log \langle m \rangle |F|)^{\theta_1} (2^{n_2} |G|)^{\theta_2} \lesssim \\ &\lesssim \sum_{m=0}^{\log \langle n \rangle} \sum_{n_1, n_2} 2^{-n_1(\frac{1}{2}-\theta_1)} 2^{-n_2(1-\theta_2)} (\log \langle n \rangle)^{1/2} (\log \langle m \rangle)^{\theta_1} |F|^{\theta_1} |G|^{\theta_2}. \end{aligned}$$

The series converges provided $\theta_1 < \frac{1}{2}$, given that all $2^{-n_1} \leq 1, 2^{-n_2} \leq 1$. Then an easy upper bound is given by

$$\left| \int_{\mathbb{R}} \sum_{I \in \mathcal{J}} r_I(t) \langle f, h_{I_n} \rangle h_I(x) g(x) dx \right| \lesssim (\log \langle n \rangle)^2 |F|^{1/p} |G|^{1/p'},$$

where $p > 2$. Then interpolation will yield the boundedness of T_5 on L^p , for $p \geq 2$. (The case $p = 2$ can be done directly). Using duality, we get that $T_5 : L^p \rightarrow L^p$ for any $1 < p < \infty$.

Remark 21. *An interesting observation is that a more careful computation (using Stirling's formula) gives a better estimate for the operatorial norm: $O((\log \langle n \rangle)^{3/2})$. This is still far from $\log \langle n \rangle$, which can be obtained through other methods. A more delicate partition and selection of the intervals \mathcal{J}_{n_1} would probably give a similar bound, but here we want to present an alternative proof based on stopping time methods, and the localization procedure.*

□

7.2 Rubio de Francia's Square Function

Let $\{[a_k, b_k]\}_{1 \leq k \leq N}$ be a family of arbitrary intervals with finite overlapping:

$$\text{i.e. } \sum_k \mathbf{1}_{[a_k, b_k]}(x) \leq B \quad \text{for any } x \in \mathbb{R}.$$

The *Rubio de Francia* square function associated to this family of intervals is a sharp cut-off square function defined by

$$Tf(x) := \left(\sum_{k=1}^N |f * \check{\mathbf{1}}_{[a_k, b_k]}(x)|^2 \right)^{1/2}.$$

Theorem 36 (Rubio de Francia). *T maps L^p into L^p boundedly for any $p \geq 2$. Moreover, the operator norm is independent on the family of intervals.*

First, one can assume without loss of generality that the family of intervals is sparse, or well distributed. That is:

$$\sum_k \mathbf{1}_{3[a_k, b_k]}(x) \leq 1 \quad \text{for any } x \in \mathbb{R}.$$

Then we proceed by linearizing the operator T :

$$Tf(x) = \sum_{k=1}^N \epsilon_k(x) (f * \check{\mathbf{1}}_{[a_k, b_k]})(x), \quad \text{where } \sum_{k=1}^N |\epsilon_k(x)|^2 \leq 1.$$

Similarly to [12], one can write

$$\mathbf{1}_{[a_k, b_k]}(\xi) \hat{f}(\xi) = \sum_{P \in \mathbb{P}_k} \langle f, \phi_P^k \rangle \hat{\phi}_P^k$$

where \mathbb{P}_k is a collection of tiles $P = I \times J$ of area $1/2$, I and J are dyadic intervals with $J \in \mathbf{J}_{[a_k, b_k]}$ and ϕ_P^k is an L^2 -normalized wave packet associated to the tile P . The set $\mathbf{J}_{[a_k, b_k]}$ represents the Whitney decomposition of $[a_k, b_k]$ with respect to its endpoints:

$J \in \mathbf{J}_{[a_k, b_k]}$ is a maximal dyadic interval contained in $[a_k, b_k]$ with the property that $\text{dist}(J, a_k) \geq |J|$ and $\text{dist}(J, b_k) \geq |J|$. In what follows, we assume the families of tiles \mathbb{P}_k to be finite, while the frequencies of the intervals are lacunary with respect to the a_k s only. Also, denote $\mathbb{P} := \cup_k \mathbb{P}_k$.

Then the model operator for the Rubio de Francia square function is

$$Tf(x) = \sum_{k=1}^N \epsilon_k(x) \sum_{P \in \mathbb{P}_k} \langle f, \phi_P^k \rangle \phi_P^k(x).$$

Remark 22. Consider the frequency interval $[0, L]$, where $L := \max_k |a_k - b_k|$. This will be our "interval of reference". Let \mathbb{P}_0 denote the collection of tiles associated to this interval. It consists of tiles

$$P = I \times \omega, \quad \text{where } I \text{ is a dyadic interval, and } \omega = \left[\frac{1}{2|I|}, \frac{1}{|I|} \right].$$

Then for any $1 \leq k \leq N$, \mathbb{P}_k can be written as the union

$$\mathbb{P}_k = \bigcup_{i \in \{-1, 0, 1\}} (\mathbb{P}_0 + \nu_{k,i}) \cap \mathcal{D}^i \cap \mathbb{P}_k$$

Here \mathcal{D}^0 represents the dyadic mesh, while \mathcal{D}^1 and \mathcal{D}^{-1} are translations half units to the right and to the left respectively. We want \mathbb{P}_k to be a subcollection of a dyadic translation of \mathbb{P}_0 , but since this is not possible, we need to include the above partitioning. Then we can assume without loss of generality that $\mathbb{P}_k = (\mathbb{P}_0 + \nu_k) \cap \mathbb{P}_k$. This will be useful later, when we consider tree structures of the tiles.

Definition 28. A subcollection T of tiles is called a *tree with top P_T* if there exists a tile $P_T = I_T \times \omega_T$ and a frequency point $\xi_T \in \omega_T$ with the property that

$$I_P \subseteq I_T, \quad \text{and} \quad \xi_T \in 7\omega_P \quad \text{for every } P \in T.$$

Remark 23. In \mathbb{P}_0 , the frequency intervals are lacunary with respect to 0.

Definition 29. Let $\{\mathbb{P}_k\}_k$ be as above. We say that $\vec{T} \subset \mathbb{P} = \bigcup_k \mathbb{P}_k$ is a *vectorial tree* if there exists a tree $T \subset \mathbb{P}_0$ so that for every $1 \leq k \leq N$,

$$\vec{T}_k := \vec{T} \cap \mathbb{P}_k = \nu_k + T \quad \text{is a frequency translation of } T.$$

Remark 24. Every \vec{T}_k is a tree in itself, with the same top interval I_T and lacunary with respect to ν_k .

Now we need to adapt the classical "sizes" to the particularity of this question. With this purpose, we define the following:

Definition 30 (Vectorial size).

$$\overrightarrow{\text{size}}_{\mathbb{P}}(f) := \sup_{\substack{\vec{T} \subset \mathbb{P} \\ \text{vectorial tree}}} \left(\frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 \right)^{1/2}$$

Definition 31 (Dual size). *This is identical to the size used in 27, for the square functions:*

$$\widetilde{\text{size}}_{\mathbb{P}}(g) := \sup_{I' \in \mathcal{J}_{\mathbb{P}}^+(I_T)} \frac{1}{|3I'|} \int_{\mathbb{R}} |g(x)| \tilde{\chi}_{3I'}(x) dx.$$

Here, given a collection \mathbb{P} of tiles, we denote by $\mathcal{J}_{\mathbb{P}}^+$ the collection of dyadic intervals I' which contain some I_P , with $P \in \mathbb{P}$.

Lemma 12 (Localization Lemma). *Let $\vec{T} \subset \mathbb{P}$ be a vectorial tree. Then*

$$\left| \int_{\mathbb{R}} \sum_{k=1}^N \epsilon_k(x) \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx \right| \lesssim \widetilde{\text{size}}_{\vec{T}}(g) \cdot \overrightarrow{\text{size}}_{\vec{T}}(f) \cdot |I_T|.$$

Proof. We consider the collection \mathcal{J} of maximal dyadic intervals J with the property that $I_P \not\subset 3J$ for all tiles $P \in \vec{T}$. Then \mathcal{J} form a partition of \mathbb{R} and thus

$$\begin{aligned} \int_{\mathbb{R}} \sum_k \epsilon_k(x) \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx &= \sum_{J \in \mathcal{J}} \int_J \sum_k \epsilon_k(x) \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx = \\ &= \sum_{J \in \mathcal{J}} \int_J \sum_k \epsilon_k(x) \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx + \end{aligned} \quad (\text{I})$$

$$+ \sum_{J \in \mathcal{J}} \int_J \sum_k \epsilon_k(x) \sum_{\substack{P \in \vec{T}_k \\ |I_P| \leq |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx \quad (\text{II})$$

We first estimate (I). Let $J \in \mathcal{J}$. In this case, the spatial intervals of the tiles are so that $|I_P| > |J|$, and no $I_P \subset 3J$, but there exists a tile $P(J) \in \mathbb{P}$ so that $I_{P(J)} \subset 3\tilde{J}$ (recall that \tilde{J} denotes the dyadic parent of J). Then $I(J)$ is contained in one of \tilde{J} , or its left or right translation, which we denote $I' \in \mathbb{P}^+(I_T)$. This means $I(J) \subseteq I'$, and $J \subset 3I'$. Also, all the intervals J contributing to (I) are so that

$J \subset 7I_T$. Fix $J \in \mathcal{J}$.

$$\begin{aligned}
\left| \int_J \sum_k \epsilon_k(x) \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx \right| &\leq \int_{\mathbb{R}} \left(\sum_k \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \right|^2 \right)^{1/2} |g(x)| \mathbf{1}_J(x) dx \lesssim \\
&\lesssim \int_{\mathbb{R}} |g(x)| \cdot \tilde{\chi}_{3I'} dx \cdot \sup_{x \in J} \left(\sum_{k=1}^N \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P(x) \right|^2 \right)^{1/2} \lesssim \\
&\lesssim \frac{1}{|3I'|} \int_{\mathbb{R}} |g(x)| \cdot \tilde{\chi}_{3I'}(x) dx \cdot \sup_{x \in J} \left(\sum_{k=1}^N \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P(x) \right|^2 \right)^{1/2} \cdot |J|.
\end{aligned}$$

Now we claim that the quantity $\sup_{x \in J} \left(\sum_{k=1}^N \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P(x) \right|^2 \right)^{1/2}$ is bounded pointwise by a maximal operator.

The tiles in \mathbb{P}_k that appear in the above sum have the property that the frequency intervals ω_P are lacunary with respect to ν_k and their length is $\leq \frac{1}{2}|J|^{-1}$. Hence they are contained in an interval of length $|J|^{-1}$ with endpoint ν_k . Let ψ_k be a bump function so that $\hat{\psi}_k \equiv 1$ on this interval and so that it is supported on a slightly larger interval. Then

$$\sum_{P \in \vec{T}_k, |I_P| > |J|} \langle f, \phi_P^k \rangle \phi_P^k(x) = \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right) * \psi_k(x)$$

and we have the pointwise estimate

$$\begin{aligned}
\left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \right| &\lesssim \sup_{K \supseteq J} \frac{1}{|K|} \int_K \left| \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k(z) \right| dz \lesssim \\
&\lesssim \inf_{y \in J} \mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y).
\end{aligned}$$

This observation leads to the estimates

$$\sup_{x \in J} \left(\sum_{k=1}^N \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P(x) \right|^2 \right)^{1/2} \lesssim \inf_{y \in J} \left(\sum_k |\mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y)|^2 \right)^{1/2} \quad (7.7)$$

$$\sup_{x \in J} \left(\sum_{k=1}^N \left| \sum_{\substack{P \in \vec{T}_k \\ |I_P| > |J|}} \langle f, \phi_P^k \rangle \phi_P(x) \right|^2 \right)^{1/2} \cdot |J| \lesssim \int_J \left(\sum_k |\mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y)|^2 \right)^{1/2} dy. \quad (7.8)$$

The estimate for (I) becomes

$$\begin{aligned} |I| &\leq \sum_{\substack{J \in \mathcal{J} \\ J \subset \gamma I_T}} \widetilde{\text{size}}(g) \cdot \int_J \left(\sum_{k=1}^N |\mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y)|^2 \right)^{1/2} dy \leq \\ &\leq \widetilde{\text{size}}_{\vec{T}}(g) \cdot \int_{\gamma I_T} \left(\sum_{k=1}^N |\mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y)|^2 \right)^{1/2} dy \lesssim \\ &\lesssim \widetilde{\text{size}}_{\vec{T}}(g) \cdot |I_T|^{1/2} \left\| \left(\sum_{k=1}^N |\mathcal{M} \left(\sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right)(y)|^2 \right)^{1/2} \right\|_2 \lesssim \\ &\lesssim \widetilde{\text{size}}_{\vec{T}}(g) \cdot |I_T|^{1/2} \left\| \sum_{k=1}^N \left| \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k \right|(y) \right\|_2 \lesssim \\ &\lesssim \widetilde{\text{size}}_{\vec{T}}(g) \cdot \text{size}_{\vec{T}}(f) \cdot |I_T|. \end{aligned}$$

Estimating (II) is going to be slightly different. In this case, for a fixed interval $J \in \mathcal{J}$, we are considering the tiles $P \in \vec{T}_k$ with $|I_P| \leq |J|$. Because $I_P \not\subset 3J$,

$$\frac{\text{dist}(J, I_P)}{|I_P|} \geq 1 \quad \text{for all } P \in \vec{T}.$$

Recall that $\vec{T}_k = T + \nu_k$, so $\phi_P^k(x) = e^{2\pi i x \nu_k} \phi_P(x)$, where P is a tile in $T \subset \mathbb{P}_0$.

$$\begin{aligned}
& \left| \sum_{J \in \mathcal{J}} \int_J \left(\sum_{k=1}^N \epsilon_k(x) \sum_{\substack{P \in \vec{T}_k \\ |I_P| < |J|}} \langle f, \phi_P^k \rangle \phi_P^k(x) \right) \cdot \bar{g}(x) dx \right| \leq \\
& \leq \sum_{J \in \mathcal{J}} \left| \int_J \sum_{\substack{P \in T \\ |I_P| < |J|}} \left(\sum_k \epsilon_k(x) \langle f, \phi_P^k \rangle \phi_P(x) e^{2\pi i \nu_k x} \right) \cdot \bar{g}(x) dx \right| \leq \\
& \leq \sum_{J \in \mathcal{J}} \sum_{\substack{P \in T \\ |I_P| < |J|}} \int_J \left(\sum_{k=1}^N |\langle f, \phi_P^k \rangle|^2 \right)^{1/2} \frac{1}{|I_P|^{1/2}} \cdot \tilde{\chi}_{I_P}^{-(M+100)} \cdot |g(x)| dx \lesssim \\
& \lesssim \sum_{J \in \mathcal{J}} \sup_{\substack{P \in T \\ |I_P| < |J|}} \left(\frac{1}{|I_P|} \sum_k |\langle f, \phi_P^k \rangle|^2 \right)^{1/2} \cdot \sup_{\substack{P \in T \\ |I_P| < |J|}} \left(\frac{1}{|I_P|} \int_{\mathbb{R}} \tilde{\chi}_{I_P}^M(x) |g(x)| dx \right) \cdot \\
& \sum_{\substack{P \in T \\ |I_P| < |J|}} |I_P| \left(1 + \frac{\text{dist}(J, I_P)}{|I_P|} \right)^{-100} \lesssim \\
& \lesssim \widetilde{\text{size}}_{\vec{T}}(g) \cdot \vec{\text{size}}_{\vec{T}}(f) \cdot \sum_{J \in \mathcal{J}} \sum_{\substack{P \in T \\ |I_P| < |J|}} |I_P| \left(1 + \frac{\text{dist}(J, I_P)}{|I_P|} \right)^{-100}
\end{aligned}$$

The estimate for (II) reduces to finding an upper bound for the last sum:

$$\begin{aligned}
& \sum_{J \in \mathcal{J}} \sum_{\substack{P \in T \\ |I_P| < |J|}} |I_P| \left(1 + \frac{\text{dist}(J, I_P)}{|I_P|} \right)^{-100} = \\
& \sum_{J \in \mathcal{J}} \sum_{k: 2^k \leq \min(|J|, |I_T|)} \sum_{\substack{|I_P| = 2^k \\ P \in T}} 2^k \left(1 + \frac{\text{dist}(J, I_P)}{|I_P|} \right)^{-100} \lesssim \\
& \lesssim \sum_{J \in \mathcal{J}} \min(|J|, |I_T|) \cdot \left(\frac{1 + \text{dist}(I_T, J)}{|I_T|} \right)^{-50} \lesssim \\
& \lesssim \sum_{\substack{J \in \mathcal{J} \\ |J| < |I_T|}} |J| \cdot \left(\frac{1 + \text{dist}(I_T, J)}{|I_T|} \right)^{-50} + \sum_{\substack{J \in \mathcal{J} \\ |J| \geq |I_T|}} |J| \cdot \left(\frac{1 + \text{dist}(I_T, J)}{|I_T|} \right)^{-50} := II_A + II_B
\end{aligned}$$

The first term is bounded by $C|I_T|$ because the intervals J are disjoint and close to I_T in this case. The second term is also bounded by I_T , because the intervals J are further and further away from I_T (recall $I_T \not\subset 3J$). This ends the proof of the localization lemma. \square

Proposition 24 (Estimates for size $\vec{\mathbb{P}}(f)$). *Let $f \in L^2_{loc}$. Then*

$$\text{size}_{\vec{\mathbb{P}}}(f) \lesssim \sup_{P \in \vec{\mathbb{P}}} \left(\frac{1}{|I_P|} \int_{\mathbb{R}} |f(x)|^2 \tilde{\chi}_{I_P}^{100}(x) dx \right)^{1/2}.$$

Proof. Let \vec{T} be a vectorial tree. It will be sufficient to show

$$\frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 \lesssim \frac{1}{|I_T|} \int_{\mathbb{R}} |f(x)|^2 \tilde{\chi}_{I_T}^{100}(x) dx.$$

First, assume that f is supported inside $5I_T$ and note

$$\frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 = \sum_{k=1}^N \left(\frac{1}{|I_T|^{1/2}} \left\| \left(\sum_{P \in \vec{T}_k} \frac{|\langle f, \phi_P^k \rangle|^2}{|I_P|} \mathbf{1}_{I_P} \right)^{1/2} \right\|_2 \right)^2.$$

In the interior we recognize modulated square functions:

$$\left\| \left(\sum_{P \in \vec{T}_k} \frac{|\langle f, \phi_P^k \rangle|^2}{|I_P|} \mathbf{1}_{I_P} \right)^{1/2} \right\|_2 = \left\| \left(\sum_{P \in T} \frac{|\langle f^k, \phi_P \rangle|^2}{|I_P|} \mathbf{1}_{I_P} \right)^{1/2} \right\|_2$$

where $f^k := e^{-2\pi i v_k x} (f * \check{\mathbf{1}}_{[a_k, b_k]})$. For these we have

$$\left\| \left(\sum_{P \in T} \frac{|\langle f^k, \phi_P \rangle|^2}{|I_P|} \mathbf{1}_{I_P} \right)^{1/2} \right\|_2 \lesssim \|f^k\|_2.$$

Unwinding the notation, we have

$$\frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 \lesssim \frac{1}{|I_T|} \sum_{k=1}^N \|f^k\|_2^2 \lesssim \frac{1}{|I_T|} \|f\|_2^2.$$

But since f is assumed to be supported inside $5I_T$, $\|f\|_2 \lesssim \|f \cdot \tilde{\chi}_{I_T}^{50}\|_2$ and the desired estimate follows.

Next, we deal with the case $\text{supp } f \subset (5I_T)^c$. Rewrite

$$\frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 = \frac{1}{|I_T|} \sum_{P \in T} \sum_{k=1}^N |\langle f, \phi_P^k \rangle|^2.$$

For a fixed $P \in T$, we have

$$\sum_{k=1}^N |\langle f, \phi_P^k \rangle|^2 \lesssim \int_{\mathbb{R}} |f(x)|^2 \cdot \left(1 + \frac{\text{dist}(x, I_P)}{|I_P|}\right)^{-M} dx.$$

Because f is supported outside $5I_T$,

$$1 + \frac{\text{dist}(x, I_P)}{|I_P|} \sim \frac{\text{dist}(x, I_P)}{|I_P|} = \frac{\text{dist}(x, I_P)}{|I_T|} \cdot \frac{|I_T|}{|I_P|}.$$

Then we have

$$\begin{aligned} \frac{1}{|I_T|} \sum_{k=1}^N \sum_{P \in \vec{T}_k} |\langle f, \phi_P^k \rangle|^2 &\lesssim \frac{1}{|I_T|} \sum_{l \geq 0} \sum_{\substack{P \in T \\ |I_P| = 2^{-l}|I_T|}} \int_{\mathbb{R}} |f(x)|^2 \left(\frac{\text{dist}(x, I_P)}{|I_T|} \cdot \frac{|I_T|}{|I_P|} \right)^{-M} dx \lesssim \\ &\lesssim \frac{1}{|I_T|} \sum_{l \geq 0} 2^l \cdot 2^{-Ml} \int_{\mathbb{R}} |f(x)|^2 \left(\frac{\text{dist}(x, I_P)}{|I_T|} \right)^{-M} dx \lesssim \\ &\lesssim \frac{1}{|I_T|} \int_{\mathbb{R}} |f(x)|^2 \cdot \tilde{\chi}_{I_T}^{100}(x) dx. \end{aligned}$$

□

Lemma 13 (Decomposition lemma for $\text{size}_{\mathbb{P}}(f)$). Assume \mathbb{P} is a collection of tiles with the property that $\text{size}_{\mathbb{P}}(f) \leq \lambda$. Then one can decompose $\mathbb{P} = \mathbb{P}' \cup \mathbb{P}''$, where $\text{size}_{\mathbb{P}'}(f) \leq \frac{\lambda}{2}$ and \mathbb{P}'' can be written as a union of disjoint vectorial trees $\mathbb{P}'' = \bigcup_{\vec{T} \in \mathbb{T}} \vec{T}$ with $\sum_{\vec{T} \in \mathbb{T}} |I_T| \lesssim \lambda^{-2} \|f\|_2^2$.

Corollary 3. Let \mathbb{P} be a collection of tiles. Then one can write $\mathbb{P} = \bigcup_n \mathbb{P}_n$ where

$$\text{size}_{\mathbb{P}_n}(f) \leq \min(2^{-n}, \text{size}_{\mathbb{P}}(f)) \quad \text{and} \quad \mathbb{P}_n = \bigcup_{\vec{T} \in \mathbb{T}_n} \vec{T}, \quad \sum_{\vec{T} \in \mathbb{T}_n} |I_T| \lesssim 2^{2n} \|f\|_2^2.$$

Lemma 14 (Decomposition lemma for $\widetilde{\text{size}}_{\mathbb{P}}(g)$). Assume \mathbb{P} is a collection of tiles with the property that $\widetilde{\text{size}}_{\mathbb{P}}(g) \leq \lambda$. Then one can decompose $\mathbb{P} = \mathbb{P}' \cup \mathbb{P}''$, where $\widetilde{\text{size}}_{\mathbb{P}'}(g) \leq \frac{\lambda}{2}$ and \mathbb{P}'' can be written as a union of disjoint vectorial trees $\mathbb{P}'' = \bigcup_{\vec{T} \in \mathbb{T}} \vec{T}$ with $\sum_{\vec{T} \in \mathbb{T}} |I_T| \lesssim \lambda^{-1} \|g\|_1$.

Proof. Recall the definition $\widetilde{\text{size}}_{\mathbb{P}}(g) := \sup_{I' \in \mathcal{J}_{\mathbb{P}}^+} \frac{1}{|3I'|} \int_{\mathbb{R}} |g(x)| \tilde{\chi}_{3I'} dx$. Define $\mathbb{P}_{\text{Stock}} := \mathbb{P}$, and $\mathbb{P}'' = \emptyset$. We start by choosing a maximal dyadic interval $I' \in \mathcal{J}_{\mathbb{P}_{\text{Stock}}}^+$ with the property that it is of largest possible length, and

$$\frac{1}{|3I'|} \int_{\mathbb{R}} |g(x)| \tilde{\chi}_{3I'} dx > \frac{\lambda}{2}. \quad (7.9)$$

We construct the tree T with top $I_T := I'$ and tiles $P \in \mathbb{P}, I_P \subset I_T$. Then $\mathbb{P}'' := \mathbb{P}'' \cup \{P : P \subseteq I_T = I'\}$, set $\mathbb{P}_{\text{Stock}} := \mathbb{P}_{\text{Stock}} \setminus T$, and restart the selection algorithm.

When there are no more intervals satisfying (7.9), we move all the remaining tiles into \mathbb{P}' . And now we are left with proving $\sum_{\vec{T} \in \mathbb{T}} |I_T| \lesssim \lambda^{-1} \|g\|_1$.

Note that the condition (7.9) implies that $I_T \subseteq \{x : \mathcal{M}g > \frac{\lambda}{2}\}$. The top intervals I_T are disjoint (they are maximal dyadic intervals), hence

$$\sum_{\vec{T} \in \mathbb{T}} |I_T| \leq |\{x : \mathcal{M}g > \frac{\lambda}{2}\}| \leq \lambda^{-1} \|g\|_1.$$

□

Corollary 4. *Let \mathbb{P} be a collection of tiles. Then one can write $\mathbb{P} = \bigcup_n \mathbb{P}_n$ where*

$$\widetilde{\text{size}}_{\mathbb{P}_n}(g) \leq \min(2^{-n}, \widetilde{\text{size}}_{\mathbb{P}}(g)) \quad \text{and} \quad \mathbb{P}_n = \bigcup_{\vec{T} \in \mathbb{T}_n} \vec{T}, \quad \sum_{\vec{T} \in \mathbb{T}_n} |I_T| \lesssim 2^n \|g\|_1.$$

7.2.1 Proof of the Main Theorem 36

Proof. We will prove restricted weak type estimates for the bilinear form Λ associated with the model operator for T :

$$\Lambda(f, g) = \int_{\mathbb{R}} \sum_{k=1}^N \epsilon_k(x) \sum_{P \in \vec{T}_k} \langle f, \phi_P^k \rangle \phi_P^k(x) \bar{g}(x) dx.$$

Let F, G be two finite subsets of \mathbb{R} and f, g two functions so that $|f| \leq \mathbf{1}_F, |g| \leq \mathbf{1}_G$.

In what follows, we will show that

$$|\Lambda(f, g)| \lesssim |F|^\alpha |G|^{\alpha'} \quad \text{for any } \alpha < \frac{1}{2}.$$

Using the decomposition lemmas for $\text{size} \vec{\mathbb{P}}(f)$ and $\widetilde{\text{size}} \mathbb{P}(g)$, and also the localization lemma 12, we have

$$\begin{aligned} |\Lambda(f, g)| &\leq \sum_{2^{-n_1} \leq \text{size} \vec{\mathbb{P}}(f)} \sum_{2^{-n_2} \leq \widetilde{\text{size}} \mathbb{P}(g)} \sum_{T \in \mathbb{T}_{n_1} \cap \mathbb{T}_{n_2}} \text{size} \vec{\mathbb{P}}_{n_1}(f) \cdot \widetilde{\text{size}} \mathbb{P}_{n_2}(g) \cdot |I_T| \lesssim \\ &\lesssim \sum_{n_1, n_2} 2^{-n_1} 2^{-n_2} \left(2^{2n_1} \|f\|_2^2\right)^{\theta_1} \cdot (2^{n_2} \|g\|_1)^{\theta_2} = \\ &= \sum_{n_1, n_2} 2^{-n_1(1-2\theta_1)} 2^{-n_2(1-\theta_2)} \|f\|_2^{2\theta_1} \|g\|_1^{\theta_2} \end{aligned}$$

In the above equation, θ_1 and θ_2 are so that $\theta_1 + \theta_2 = 1$. For the above series to converge, we need $\theta_1 < \frac{1}{2}$. Given that the functions f and g are so that $|f| \leq \mathbf{1}_F, |g| \leq \mathbf{1}_G$, the sizes are $\lesssim 1$, and hence

$$|\Lambda(f, g)| \lesssim |F|^{\theta_1} |G|^{\theta_2}.$$

The L^p boundedness for T follows from interpolation results. □

7.3 Vector-Valued Carleson Operator

In this section, we give another proof for the boundedness of the Carleson operator taking values in l^r spaces. This question was answered in [22], in the more general context of UMD spaces.

The continuous Carleson operator is given by

$$C_{\mathbb{R}} f(x) = \sup_N \left| \int_{\xi < N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \quad (7.10)$$

and its study can be reduced to the discretized version

$$C_{\mathbb{P}}f(x) = \sum_{P \in \mathbb{P}} \langle f, \phi_{P_1} \rangle \phi_{P_1}(x) \mathbf{1}_{\{x: N(x) \in \omega_2\}}.$$

Here \mathbb{P} is a collection of bi-tiles $P = (I_P \times \omega_1, I_P \times \omega_2)$, and $N(x)$ is an arbitrary measurable function which maximizes 7.10. The operator $C_{\mathbb{P}}$ is bounded on L^p for any $1 < p < \infty$. Here we prove a vector-valued version of this result:

Theorem 37.

$$\left\| \left(\sum_k |C_{\mathbb{P}}f_k|^r \right)^{1/r} \right\|_p \lesssim \left\| \left(\sum_k |f_k|^r \right)^{1/r} \right\|_p \quad (7.11)$$

for any $1 < p < \infty$, $1 < r < \infty$.

We follow the presentation in [6], which is based on the concepts of sizes and energies. For the function f , the notions of size and energy are identical with those define in section 2.4.2. For the second entry, they are more complicated, and the main reason is the characteristic function $\mathbf{1}_{\{x: N(x) \in \omega_2\}}$ appearing in the definition of $C_{\mathbb{P}}$. There are still good estimates for energies and sizes, and we present their localized versions, which will be useful for our local Carleson operator:

Proposition 25. *Let I be a dyadic interval, and \mathbb{P} a collection of bi-tiles. Let g be a function so that $|g| \leq \mathbf{1}_E$. Then:*

$$\text{size}_{\mathbb{P}(I)}(\langle \phi_{P_1} \mathbf{1}_{\omega_2} \circ N, g \rangle) \lesssim \sup_{P \in \mathbb{P}(I)} \sup_{\substack{P' \supseteq P \\ I_{P'} \subset 10I_P}} \frac{1}{|I_{P'}|} \int_{\mathbb{R}} \mathbf{1}_E \tilde{\chi}_{I_{P'}}^M dx$$

$$\text{energy}_{\mathbb{P}(I)}(\langle \phi_{P_1} \mathbf{1}_{\omega_2} \circ N, g \rangle) \lesssim \|f \cdot \tilde{\chi}_I\|_1.$$

Moreover, if the collection of bitiles is sparse, we can estimate

$$\text{size}_{\mathbb{P}(I)}(\langle \phi_{P_1} \mathbf{1}_{\omega_2} \circ N, g \rangle) \lesssim \widetilde{\text{size}_{\mathbb{P}(I)}} \mathbf{1}_E.$$

Lemma 15 (Prop. 7. 7 in [6]). *Let \mathbb{P} be a finite collection of bi-tiles and f and g measurable functions. Then*

$$\left| \int_{\mathbb{R}} C_{\mathbb{P}} f(x) g(x) dx \right| \lesssim (\text{size}(\langle f, \phi_{P_1} \rangle))^{\theta_1} (\text{size}(\langle \phi_{P_1} \mathbf{1}_{\omega_2} \circ N, g \rangle))^{\theta_2} \\ (\text{energy}(\langle f, \phi_{P_1} \rangle))^{1-\theta_1} (\text{energy}(\langle \phi_{P_1} \mathbf{1}_{\omega_2} \circ N, g \rangle))^{1-\theta_2}$$

for any $0 \leq \theta_1 < 1, 0 < \theta_2 \leq 1/2$ with $\theta_1 + 2\theta_2 = 1$.

Proposition 26 (Localized $C_{\mathbb{P}}$). *Let I_0 be a dyadic interval, and F, G sets of finite measure. Then for any $1 < p < \infty$*

$$\left| \int_{\mathbb{R}} C_{\mathbb{P}(I_0)}(f \cdot \mathbf{1}_F)(x) (g \cdot \mathbf{1}_G)(x) dx \right| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G \right)^{\frac{1}{p}} \|f \cdot \tilde{\chi}_{I_0}\|_p \|g \cdot \tilde{\chi}_{I_0}\|_{p'}.$$

Proof. The proof is standard by now: prove generalized restricted type estimates. Let E_1, E_2 be sets of finite measure, the exceptional set is defined as $\tilde{\Omega} = \{x : \mathcal{M} \mathbf{1}_{E_1} > C \frac{|E_1|}{|E_2|}\}$ and $E'_2 = E_2 \setminus \tilde{\Omega}$. Let $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E'_2}$, and look only at the tiles $P \in \mathbb{P}(I)$ with the property that $1 + \frac{\text{dist}(I_P, \tilde{\Omega}^c)}{|I_P|} \sim 2^d$. Here we present the case $d \geq 2$ because in this case the intervals I_P are essentially disjoint and the collection of tiles is sparse. In the case d small, the proof follows the same ideas but becomes more technical because of the more complicated sizes.

Also, for $d \geq 2$, the construction of the collections $\mathcal{J}_1^{n_1}$ is less elaborated:

- for any n_1 with $2^{-n_1} \leq 2^d \frac{|E_1|}{|E_2|}$, $\mathcal{J}_1^{n_1}$ will contain the maximal dyadic intervals I with the property that

$$\widetilde{\text{size}}_{\mathbb{P}_{n_1}(I)}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F) \sim \frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \mathbf{1}_F \cdot \tilde{\chi}_I^M dx \sim 2^{-n_1}.$$

- when I above is intersected with another interval $I' \in \mathcal{J}_2^{n_2}$,

$$\widetilde{\text{size}}_{\mathbb{P}(I \cap I')}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F), \quad \frac{1}{|I \cap I'|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \mathbf{1}_F \cdot \tilde{\chi}_{I \cap I'}^M dx \lesssim \widetilde{\text{size}}_{I_0}(\mathbf{1}_{E_1} \cdot \mathbf{1}_F).$$

We have:

$$\begin{aligned}
& \left| \sum_{P \in P^d(I_0)} \langle f \cdot \mathbf{1}_F, \phi_{P_1} \rangle \langle \phi_{P_1} \cdot \mathbf{1}_{\omega_2} \circ N, g \cdot \mathbf{1}_G \rangle \right| \lesssim \\
& \lesssim \sum_{n_1, n_2} \sum_{I \in \mathcal{J}^{n_1, n_2}} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \right)^{\theta_1} \left(\widetilde{\text{size}}_{\mathbb{P}(I)} \mathbf{1}_{E_2} \cdot \mathbf{1}_G \right)^{\theta_2} \cdot \\
& \quad \cdot \left(\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \tilde{\chi}_I dx \right)^{\frac{1-\theta_1}{2}} \cdot \left(\frac{1}{|I|} \int_{\mathbb{R}} \mathbf{1}_{E_2} \cdot \mathbf{1}_G \tilde{\chi}_I dx \right)^{1-\theta_2} \cdot |I| \lesssim \\
& \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{E_1} \cdot \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{p}} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{E_2} \cdot \mathbf{1}_G \right)^{\frac{1}{p}} |E_1|^{1/p} |E_2|^{1/p'} 2^{-\tilde{M}d}
\end{aligned}$$

The restriction on p is that $\frac{1}{p} \leq \frac{1+\theta_1}{2}$, but since θ_1 is arbitrary, this is not really a constraint. \square

7.3.1 Proof of Theorem 37

Now we have F, G sets of finite measure, $|G| = 1$. $\Omega = \{x : \mathcal{M}\mathbf{1}_F > C|F|\}$, $G' = G \setminus \Omega$,

$$\left(\sum_k |f_k|^r \right)^{1/r} \leq \mathbf{1}_F, \quad \left(\sum_k |g_k|^{r'} \right)^{1/r'} \leq \mathbf{1}_{G'}.$$

The collection of intervals $\mathcal{J}_1^{n_1}, \mathcal{J}_2^{n_2}$ are constructed as before.

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \sum_k C_{\mathbb{P}}(f_k)(x) \cdot g_k(x) dx \right| \lesssim \sum_{n_1, n_2} \sum_{I \in \mathcal{J}^{n_1, n_2}} \sum_k \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F \right)^{\frac{1+\theta_1}{2} - \frac{1}{r}} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{G'} \right)^{\frac{1}{r}} \|f_k \cdot \tilde{\chi}_I\|_r \|g_k \cdot \tilde{\chi}_I\|_{r'} \\
& \lesssim \sum_{n_1, n_2} 2^{-\frac{n_1}{p}} 2^{-n_2} \sum_{I \in \mathcal{J}^{n_1, n_2}} |I| \lesssim |F|^{\frac{1}{p}} \cdot 2^{-\tilde{M}d}.
\end{aligned}$$

Remark 25. Similar to the case of BHT operator, we can prove a localized version of the vector-valued extension for $C_{\mathbb{P}}$ and then use that in order to prove iterated vector-valued estimates.

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